Jurnal
Teknologi

Full Paper

The Squared Commutativity Degree of Dihedral Groups

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Article history
Received
10 February 2015
Received in revised form
25 October 2015
Accepted
26 October 2015

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Graphical abstract

The Squared Commutativity Degree of a Group:

$$P^2(G) = \frac{\left|\left\{(x,y)\in G\times G : (xy)^2 = (yx)^2\right\}\right|}{|G|^2}$$

Abstract

The commutativity degree of a finite group is the probability that a random pair of elements in the group commute. Furthermore, the *n*-th power commutativity degree of a group is a generalization of the commutativity degree of a group which is defined as the probability that the *n*-th power of a random pair of elements in the group commute. In this paper, the *n*-th power commutativity degree for some dihedral groups is computed for the case *n* equal to 2, called the squared commutativity degree.

Keywords: Commutativity degree; Dihedral group; finite group

Abstrak

Darjah kekalisan tukar tertib bagi suatu kumpulan terhingga adalah kebarangkalian bahawa sepasang unsur yang dipilih secara rawak dari kumpulan tersebut adalah kalis tukar tertib. Selain itu, kuasa ke-*n* bagi darjah kekalisan tukar tertib bagi suatu kumpulan adalah pengitlakan darjah kekalisan tukar tertib bagi suatu kumpulan yang ditakrifkan sebagai kebarangkalian bahawa kuasa ke-*n* bagi sepasang unsur yang dipilih secara rawak dari kumpulan tersebut adalah kalis tukar tertib. Dalam kajian ini, kuasa ke-*n* bagi darjah kekalisan tukar tertib bagi kumpulan dwihedron ditentukan yang mana *n* bersamaan dengan 2, disebut sebagai kuasa dua darjah kekalisan tukar tertib.

Kata kunci: Darjah kekalisan tukar tertib, kumpulan Dwihedron, kumpulan terhingga

1.0 INTRODUCTION

Commutativity degree is the term that is used to determine the abelianness of groups. If G is a finite group, then the commutativity degree of G, denoted by $P(G)$, is the probability that two randomly chosen elements of G commute. The first appearance of this concept was in 1944 by Miller[1]. After a few years, the idea to compute $P(G)$ for symmetric groups has been introduced by both Erdos and Turan[2] in 1968.

Mohd Ali and Sarmin[3] in 2010 extended the definition of commutativity degree of a group and defined a new generalization of this degree which is called the *n*-th commutativity degree of a group G, $P_n(G)$ where it is equal to the probability that the *n*-th power of a random element commutes with another random element from the same group. They determined $P_n(G)$ for 2 generator 2-groups of nilpotency class two.

A few years later, Erfanian *et al.*[4] gave the relative case of *n*-th commutativity degree. They identify the probability that the *n*-th power of random element of a subgroup, *H* commutes with another random element of a group *G*, denoted as $P_n(H,G)$.

In this research, the commutativity degree is further extended by defining a concept called the probability that the *n*-th power of a random pair of elements in the group commute, denoted as $P^n(G)$. However, the focus of this research is only for the determination of $P^n(G)$, where $n = 2$ and *G* is some Dihedral groups. Here, $P^2(G)$ is called the squared commutativity degree.

2.0 PRELIMINARIES

In this section, some important definitions which are the notion of commutativity degree and its generalization is stated.

Definition 2.1[2] The Commutativity Degree of a Group G

Let *G* be a finite group. The commutativity degree of a group *G*, is given as:

$$P(G) = \frac{\left|\left\{(x,y) \in G \times G \middle| xy = yx\right\}\right|}{|G|^2}.$$

Definition 2.2[3] The *n*-th Commutativity Degree of a Group G

Let *G* be a finite group. The *n*-th commutativity degree of a group *G*, is given as:

$$D_n = \langle a,b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle.$$

Definition 2.3[5] Dihedral Groups of Degree *n*

For $n \geq 3$, D_n is denoted as the set of symmetries of a regular *n*-gon. Furthermore, the order of D_n is $2n$ or equivalently $|D_n| = 2n$. The Dihedral groups, D_n can be represented in a form of generators and relations given in the following representation:

$$D_n = \langle a,b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle.$$

Definition 2.4[6] *n*-th Centralizer of *a* in G

Let *a* be a fixed element of a group G. The *n*-th centralizer of *a* in G, $C_G(a)$ is the set of all elements in G that commute with a^n. In symbols,

$$C_G^n(a) = \left\{g \in G \middle| ga^n = a^n g\right\} = C_G(a^n).$$

Here $C_G^n(a)$ is a subgroup of G and $\bigcap_{a \in G} C_G^n(a) = C_G(G^n)$, where $G^n = \left\{a^n \middle| a \in G\right\}$. Now define $T_G^n(a) = \left\{g \in G \middle| (ga)^n = (ag)^n\right\}$ and $T_G^n(G) = \bigcap_{a \in G} T_G^n(a)$. It is easy to see that $T_G^n(a)$ may not be a subgroup of G. But it can be seen easily that $T_G^n(G) = C_G(G^n)$ and so $T_G^n(G)$ is a normal subgroup of G. To prove $T_G^n(G) \subseteq C_G(G^n)$, let $a \in T_G^n(G)$. Then for all $g \in G$, $(ag)^n = (ga)^n$.

Therefore $\left(a\left(a^{-1}g\right)\right)^n = \left(\left(a^{-1}g\right)a\right)^n$ and so $g^n = a^{-1}g^n a$. Hence $ag^n = g^n a$ and $a \in C_G(G^n)$. To see $C_G(G^n) \subseteq T_G^n(G)$, let $a \in C_G(G^n)$. Then for all $g \in G$, $(ag)^n = (ga)^n$. Therefore $a(ag)^n = (ag)^n a$ and so $(ag)^n = a^{-1}(ag)^n a$. Hence $(ag)^n (ga)^n$ and $a \in T_G^n(G)$

.

Definition 2.5 *n*-th Center of a Group

The *n*-th center $Z^n(G)$ of a group G is the *n*-th power of the set of elements in G that commute with every element of G. In symbols,

$$Z^n(G) = \{a \in G \mid (ax)^n = (xa)^n \text{ for all } x \text{ in } G\}.$$

3.0 RESULTS AND DISCUSSION

This section start by defining a new definition as follow:

Definition 3.1 The *n*-th Power Commutativity Degree

Let *G* be a finite group. The *n*-th power commutativity degree of a group *G*, is given as:

$$P^n(G) = \frac{\left|\left\{(x,y) \in G \times G : (xy)^n = (yx)^n\right\}\right|}{|G|^2}.$$

When $n = 2$, then

$$P^2(G) = \frac{\left|\left\{(x,y) \in G \times G : (xy)^2 = (yx)^2\right\}\right|}{|G|^2}$$

is called the squared commutativity degree of a group.

Next, the following propositions are given which play an important role in the proof of Theorems. Proposition 3.1 plays an important role in the proof of Theorem 3.1 as well as Proposition 3.2 in proving Theorem 3.2.

Proposition 3.1

Let G be a Dihedral group of order $2n$ where $n \geq 6$, and n is even. If $n/2$ is odd, then $\left|T_G^2(x)\right| = \frac{|G|}{2} + 2.$ Meanwhile, if $n/2$ is even, then $\left|T_G^2(x)\right| = \frac{|G|}{2} + 4$ in which $x \notin Z^2(G)$, $T_G^2(x) = \{g \in G : (gx)^2 = (xg)^2\}$ and $Z^2(G) = \left\{a \in G \mid (ax)^2 = (xa)^2\right\}$.

Proof

Case 1: $n/2$ is odd.

Let $G = D_n$ where $|D_n| = 2n$ and $n/2$ is odd. Suppose $A = \left\{e, a, \ldots, a^{n-1}\right\}$ and $B = \left\{b, ab, a^2b, \ldots, a^{n-1}b\right\}$. Recall Definition 2.3 then we have $T_G^2(e) = T_G^2(a) = \ldots = T_G^2(a^{n-1}) = G$ since for all $y, z \in A$, we also have $yz = zy$ and $(yz)^2 = (zy)^2$. Furthermore, for all $y \in A$, and $z \in G \setminus A$ we have $(yz)^2 = (zy)^2$. We also have

$$T_G^2(b) = \{e, a, a^2, \ldots, a^{n-1}, b, a^{\frac{n}{2}}b\} = T_G^2(a^{\frac{n}{2}}b)$$
$$T_G^2(ab) = \{e, a, a^2, \ldots, a^{n-1}, ab, a^{\frac{n}{2}+1}b\} = T_G^2(a^{\frac{n}{2}+1}b)$$
$$T_G^2(a^2b) = \{e, a, a^2, \ldots, a^{n-1}, a^2b, a^{\frac{n}{2}+2}b\} = T_G^2(a^{\frac{n}{2}+2}b)$$
$$\vdots$$
$$T_G^2(a^{\frac{n}{2}-1}b) = \{e, a, a^2, \ldots, a^{n-1}, a^{\frac{n}{2}-1}b, a^{n-1}b\} = T_G^2(a^{n-1}b)$$

Since for $y \in B$ and all $z \in B$, there are only two pairs of elements which satisfy $(yz)^2 = (zy)^2 = e$. Note that $Z^2(G) = \bigcap_{x \in G} T_G^2(x)$, therefore $Z^2(G) = \{e, a, a^2, \ldots, a^{n-1}\}$ implies $\left|Z^2(G)\right| = n = |G|/2$. Assume that $x \notin Z^2(G)$, therefore we have

$$\left|T_G^2(x)\right| = \left|Z^2(G)\right| + 2$$
$$= \frac{|G|}{2} + 2. \qquad \square$$

Note that, if $P = \left\{x \in Z^2(G)\right\}$, thus $|P| = \frac{|G|}{2}$. Meanwhile, if $Q = \left\{x \notin Z^2(G)\right\}$, thus $|Q| = \frac{|G|}{2}$. Therefore,

$$|P| + |Q| = \frac{|G|}{2} + \frac{|G|}{2} = |G|.$$

Case 2: $n/2$ is even.

Let $G = D_n$ where $|D_n| = 2n$ and $n/2$ is even. Suppose $A = \left\{e, a, \ldots, a^{n-1}\right\}$ and $B = \left\{b, ab, a^2b, \ldots, a^{n-1}b\right\}$. Recall Definition 2.3 then we have $T_G^2(e) = T_G^2(a) = \ldots = T_G^2(a^{n-1}) = G$ since for all $y, z \in A$, we also have $yz = zy$ and $(yz)^2 = (zy)^2$. Furthermore, for all $y \in A$, and $z \in G \setminus A$ we have $(yz)^2 = (zy)^2$. We also have

$$T_G^2(b) = \{e, a, a^2, \ldots, a^{n-1}, b, a^{\frac{n}{4}}b, a^{\frac{n}{2}}b, a^{\frac{3n}{4}}b\}$$
$$= T_G^2(a^{\frac{n}{4}}b) = T_G^2(a^{\frac{n}{2}}b) = T_G^2(a^{\frac{3n}{4}}b)$$
$$T_G^2(ab) = \{e, a, a^2, \ldots, a^{n-1}, ab, a^{\frac{n}{4}+1}b, a^{\frac{n}{2}+1}b, a^{\frac{3n}{4}+1}b\}$$
$$= T_G^2(a^{\frac{n}{4}+1}b) = T_G^2(a^{\frac{n}{2}+1}b) = T_G^2(a^{\frac{3n}{4}+1}b)$$
$$\vdots$$
$$T_G^2(a^{\frac{n}{4}-1}b) = \{e, a, a^2, \ldots, a^{\frac{n}{4}-1}b, a^{\frac{n}{2}-1}b, a^{\frac{3n}{4}-1}b, a^{n-1}b\}$$
$$= T_G^2(a^{\frac{n}{2}-1}b) = T_G^2(a^{\frac{3n}{4}-1}b) = T_G^2(a^{n-1}b)$$

Since for $y \in B$ and all $z \in B$, there are only four pairs of elements that satisfy $(yz)^2 = (zy)^2$ which contain the elements of e and $a^{n/2}$. Note that $Z^2(G) = \bigcap_{x \in G} T_G^2(x)$, $Z^2(G) = \{e, a, a^2, \ldots, a^{n-1}\}$ implies $\left|Z^2(G)\right| = n = |G|/2$. Assume that $x \notin Z^2(G)$, therefore we have

$$\left|T_G^2(x)\right| = \left|Z^2(G)\right| + 4$$
$$= \frac{|G|}{2} + 4. \qquad \square$$

Example 1:

Let $G = D_6$ and $D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Then we have,

$$T_G^2(e) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(a) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(a^2) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(a^3) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(a^4) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(a^5) = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\} = D_6$$
$$T_G^2(b) = \{e, a, a^2, a^3, a^4, a^5, b, a^3b\} = T_G^2(a^3b)$$
$$T_G^2(ab) = \{e, a, a^2, a^3, a^4, a^5, ab, a^4b\} = T_G^2(a^4b)$$
$$T_G^2(a^2b) = \{e, a, a^2, a^3, a^4, a^5, a^2b, a^5b\} = T_G^2(a^5b)$$

Here $Z^2(D_6) = \bigcap_{x \in D_6} T_{D_6}^2(x) = \{e, a, a^2, a^3, a^4, a^5\}$. For $x \notin Z^2(D_6)$,

$\left|T_{D_6}^2(x)\right| = \left|Z^2(D_6)\right| + 2$ implies that $\left|T_{D_6}^2(x)\right| = \frac{|D_6|}{2} + 2$.

Example 2:

Let $G = D_8$ and $D_6 = \{e, a, a^2, \ldots, a^7, b, ab, a^2b, \ldots, a^7b\}$ where $|D_8| = 16$ Then we have,

$$T_G^2(e)=\{e,a,a^2,\dots,a^7,b,ab,a^2b,\dots,a^7b\}=D_8$$
$$T_G^2(a)=\{e,a,a^2,\dots,a^7,b,ab,a^2b,\dots,a^7b\}=D_8$$
$$T_G^2(a^2)=\{e,a,a^2,\dots,a^7,b,ab,a^2b,\dots,a^7b\}=D_8$$
$$T_G^2(a^3)=\{e,a,a^2,\dots,a^7,b,ab,a^2b,\dots,a^7b\}=D_8$$
$$\vdots$$
$$T_G^2(a^7)=\{e,a,a^2,\dots,a^7,b,ab,a^2b,\dots,a^7b\}=D_8$$
$$T_G^2(b)=\{e,a,a^2,\dots,a^7,b,a^2b,a^4b,a^6b\}$$
$$=T_G^2(a^2b)=T_G^2(a^4b)=T_G^2(a^6b)$$
$$T_G^2(ab)=\{e,a,a^2,\dots,a^7,ab,a^3b,a^5b,a^7b\}$$
$$=T_G^2(a^3b)=T_G^2(a^5b)=T_G^2(a^7b)$$

Here $Z^2(D_8)=\bigcap_{x\in D_8}T_{D_8}^2(x)=\{e,a,a^2,a^3,a^4,a^5,a^6,a^7\}$. For $x\notin Z^2(D_8)$, $\left|T_{D_8}^2(x)\right|=\left|Z^2(D_8)\right|+4$ implies that $\left|T_{D8}^2(x)\right|=\dfrac{|D_8|}{2}+4.$

Proposition 3.2

Let G be a Dihedral group of order $2n$ where $n\geq 3$ and n is odd. Then $\left|T_G^2(x)\right|=\dfrac{|G|}{2}+1.$

Proof

Let $G=D_n$ where $|D_n|=2n$ and $n/2$ is odd. Suppose $A=\{e,a,\dots,a^{n-1}\}$ and $B=\{b,ab,a^2b,\dots,a^{n-1}b\}$. Recall Definition 2.3 then we have $T_G^2(e)=T_G^2(a)=\dots=T_G^2(a^{n-1})=G$ since for all $y,z\in A$, we also have $yz=zy$ and $(yz)^2=(zy)^2$. Furthermore, for all $y\in A$, and $z\in G\setminus A$ we have $(yz)^2=(zy)^2$. We also have

$$T_G^2(b)=\{e,a,a^2,\dots,a^{n-1},b\},T_G^2(ab)=\{e,a,a^2,\dots,a^{n-1},ab\},$$
$$T_G^2(a^2b)=\{e,a,a^2,\dots,a^{n-1},a^2b\},\cdots,T_G^2(a^{\frac{n}{2}-1}b)=\{e,a,a^2,\dots,a^{n-1},a^{\frac{n}{2}-1}b\}.$$

Since for $y\in B$ and all $z\in B$, there is only one pair of elements which satisfy $(yz)^2=(zy)^2=e$. Note that $Z^2(G)=\bigcap_{x\in G}T_G^2(x)$, $Z^2(G)=\{e,a,a^2,\dots,a^{n-1}\}$ implies $\left|Z^2(G)\right|=n=|G|/2$. Assume that $x\notin Z^2(G)$, therefore we have

$$\left|T_G^2(x)\right|=\left|Z^2(G)\right|+1$$
$$=\frac{|G|}{2}+1.$$
∎

Example 3:

Let $G=D_3$ and $D_3=\{e,a,a^2,b,ab,a^2b\}$. Then we have,

$$T_G^2(e)=\{e,a,a^2,b,ab,a^2b\}=D_3$$
$$T_G^2(a)=\{e,a,a^2,b,ab,a^2b\}=D_3$$
$$T_G^2(a^2)=\{e,a,a^2,b,ab,a^2b\}=D_3$$
$$T_G^2(b)=\{e,a,a^2,b\}$$
$$T_G^2(ab)=\{e,a,a^2,ab\}$$
$$T_G^2(a^2b)=\{e,a,a^2,a^2b\}$$

Here $Z^2(D_3)=\bigcap_{x\in D_3}T_{D_3}^2(x)=\{e,a,a^2\}$. For $x\notin Z^2(D_3)$, $|T_{D_3}^2|=|Z^2(D_3)|+1$ implies that $|T_{D_3}^2|=\dfrac{|D_3|}{2}+1$.

The main results of this research are stated in the following two theorems.

Theorem 3.1

Let G be Dihedral groups of order $2n$ where $n\geq 6$ and n is even.

i. If $\dfrac{n}{2}$ is odd then $P^2(G)=\dfrac{4\left|Z^2(G)\right|+|G|+4}{4|G|}$.

ii. If $\dfrac{n}{2}$ is even then $P^2(G)=\dfrac{4\left|Z^2(G)\right|+|G|+8}{4|G|}$.

Proof

By Definition 2.6, we have

$$P^2(G)=\frac{\left|\{(x,y)\in G\times G:(xy)^2=(yx)^2\}\right|}{|G|^2}$$
$$=\frac{1}{|G|^2}\sum_{x\in G}\left|\{y\in G\,|\,(xy)^2=(yx)^2\}\right|$$
$$=\frac{1}{|G|^2}\sum_{x\in G}\left|T_G^2(x)\right|$$
$$=\frac{1}{|G|^2}\left[\sum_{x\in Z^2(G)}\left|T_G^2(x)\right|+\sum_{x\notin Z^2(G)}\left|T_G^2(x)\right|\right]$$
$$=\frac{1}{|G|^2}\left[\left|Z^2(G)\right||G|+\sum_{x\notin Z^2(G)}\left|T_G^2(x)\right|\right].$$

i. By Proposition 3.1 (for $n/2$ is odd),

$$P^2(G) = \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|}{2}\left(\frac{|G|}{2}+2\right)\right]$$
$$= \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|^2}{4} + |G|\right]$$
$$= \frac{|Z^2(G)|}{|G|} + \frac{1}{|G|} + \frac{1}{4}$$
$$= \frac{|Z^2(G)|+1}{|G|} + \frac{1}{4}$$
$$= \frac{4|Z^2(G)| + |G| + 4}{4|G|}. \qquad \square$$

ii. By Proposition 3.1 (for $n/2$ is even),

$$P^2(G) = \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|}{2}\left(\frac{|G|}{2}+4\right)\right]$$
$$= \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|^2}{4} + 2|G|\right]$$
$$= \frac{|Z^2(G)|}{|G|} + \frac{1}{4} + \frac{2}{|G|}$$
$$= \frac{|Z^2(G)|+2}{|G|} + \frac{1}{4}$$
$$= \frac{4|Z^2(G)| + |G| + 8}{4|G|}. \qquad \square$$

Theorem 3.2
Let G be Dihedral groups of order $2n$ where $n \geq 3$. If n is odd then $P^2(G) = \dfrac{4|Z^2(G)| + |G| + 2}{4|G|}$.

Proof
By Definition 2.6, we have

$$P^2(G) = \frac{|\{(x,y) \in G \times G : (xy)^2 = (yx)^2\}|}{|G|^2}$$
$$= \frac{1}{|G|^2}\sum_{x \in G}|\{y \in G | (xy)^2 = (yx)^2\}|$$
$$= \frac{1}{|G|^2}\sum_{x \in G}|T_G^2(x)|$$
$$= \frac{1}{|G|^2}\left[\sum_{x \in Z^2(G)}|T_G^2(x)| + \sum_{x \notin Z^2(G)}|T_G^2(x)|\right]$$
$$= \frac{1}{|G|^2}\left[|Z^2(G)||G| + \sum_{x \notin Z^2(G)}|T_G^2(x)|\right].$$

By Proposition 3.2,

$$P^2(G) = \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|}{2}\left(\frac{|G|}{2}+1\right)\right]$$
$$= \frac{1}{|G|^2}\left[|Z^2(G)||G| + \frac{|G|^2}{4} + \frac{|G|}{2}\right]$$
$$= \frac{|Z^2(G)|}{|G|} + \frac{1}{4} + \frac{1}{2|G|}$$
$$= \frac{|Z^2(G)| + \frac{1}{2}}{|G|} + \frac{1}{4}$$
$$= \frac{4|Z^2(G)| + |G| + 2}{4|G|}. \qquad \square$$

4.0 CONCLUSION

The research determined the squared commutativity degree of Dihedral groups. The results have been found for n even and n odd. However, for n even, the result has been divided into two cases where $n/2$ is odd and $n/2$ is even.

Acknowledgement

The authors would like to acknowledge Ministry of Education (MOE) Malaysia and Research Management Centre (RMC), Universiti Teknologi Malaysia for the financial funding through the Research University Grant (RUG) Vote No 08H07 and No 06H18. The first author would also like to thank Ministry of Education (MOE) for her MyPhD scholarship.

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