Seventh International Conference on Control, Automation, Robotics And Vision (ICARCV'02), Dec 2002, Singapore

Proportional-Integral Sliding Mode Tracking Controller with Application to a Robot Manipulator

MOHAMAD NOH AHMAD, JOHARI H. S. OSMAN

Dept. of Robotics and Mechatronics, Faculty of Electrical Engineering, Universiti Teknologi Malaysia, 81310 UTM Skudai, Malaysia. e-mail: noh@suria.fke.utm.my

MOHD RUDDIN A. GHANI

Kolej Universiti Teknikal Kebangsaan Malaysia, 75450 Ayer Keroh, Melaka, Malaysia.

Abstract

This paper presents the development of a Proportional-Integral sliding mode controller to control a class of uncertain systems. It is assumed that the plant to be controlled can be represented by its nominal and bounded parametric uncertainties. A robust sliding mode controller is newly derived so that the actual trajectory tracks the desired trajectory as closely as possible despite the non-linearities and input couplings present in the system. The Proportional-Integral sliding mode is chosen to ensure the stability of overall dynamics during the entire period i.e. the reaching phase and the sliding phase. The controller is applied to the control of a two-link planar robot manipulator.

1 Introduction

Variable structure control with Sliding Mode Control (SMC) has been widely applied to system with uncertainties and/or input couplings [1], [2]. The design philosophy behind the SMC is to obtain a high-speed switching control law to drive the nonlinear plant's state trajectory onto a specified and user-chosen surface called the sliding or switching surface. When a system is in the sliding mode, its dynamics is strictly determined by the dynamics of the sliding surfaces and hence insensitive to parameter variations and system disturbances. Nevertheless, the system posses no such insensitivity property during the reaching phase. Therefore insensitivity cannot be ensured throughout the entire response and the robustness during the reaching phase is normally improved by high-gain feedback control [3].

Recently, a variety of the SMC known as Integral Sliding Mode Control (ISMC) has surfaced in the literature [4], [5], [6]. Different from the conventional SMC design approaches, the order of the motion equation in ISMC is equal to the order of the original system, rather than reduced by the number of dimension of the control input. This is established by making use of the integral type switching surface. With this approach, the robustness of the system can be guaranteed throughout the entire response of the system starting from the initial time instance.

In this paper, the problem of robust tracking for a class of dynamical systems with uncertainties is considered. On the basis of sliding mode control theory, a class of variable structure controllers for robust tracking of dynamical signals is proposed. It is shown theoretically that for system with matched uncertainties, the tracking error is guaranteed to decrease asymptotically to zero. In fact the system dynamics during the sliding phase can easily be shaped up using any conventional pole placement method.

2 Problem Formulation

Consider an uncertain system described by

$$\dot{X}(t) = [A + \Delta A(t)]X(t) + [B + \Delta B(t)]u(t) \qquad (1)$$

where $X(t) \in R^n, u(t) \in R^m$, represent the state and input vectors, respectively. A and B are constant matrices of appropriate dimensions.

Define the state vector of the system as

$$X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \qquad (2)$$

Let a continuous function $X_d(t) \in R^n$ be the desired state trajectory, where $X_d(t)$ is defined as:

$$X_d(t) = [x_{d1}(t), x_{d2}(t), \ldots, x_{dn}(t)]^T \qquad (3)$$

Define the tracking error, $Z(t)$ as

$$Z(t) = X(t) - X_d(t) \qquad (4)$$

In this study, the following assumptions are made:

i) The state vector $X(t)$ can be fully observed;

ii) There exist continuous functions $H(t)$ and $E(t)$ such that for all $X(t) \in R^n$ and all t:

$$\begin{aligned} \Delta A(t) &= BH(t); \quad \|H(t)\| \le \alpha \\ \Delta B(t) &= BE(t); \quad \|E(t)\| \le \beta \end{aligned} \qquad (5)$$

iii) There exist a Lebesgue function $\Omega(t) \in R$, which is integrable on bounded interval such that

$$\dot{X}_d(t) = AX_d(t) + B\Omega(t) \qquad (6)$$

iv) The pair (A, B) is controllable.

In view of equations (4), (5) and (6), equation (1) can be written as

$$\begin{aligned} \dot{Z}(t) = {} & [A + BH(t)]Z(t) + BH(t)X_d(t) \\ & - B\Omega(t) + [B + BE(t)]u(t) \end{aligned} \qquad (7)$$

Define the Proportional-Integral sliding surface as

$$\sigma(t) = CZ(t) - \int_0^t [CA + CBK]Z(\tau)d\tau \qquad (8)$$

where $C \in R^{m\times n}$ and $K \in R^{m\times n}$ are constant matrices. The matrix K satisfies

$$\lambda_{max}(A+BK) < 0 \quad (9)$$

and C is chosen such that $CB \in R^{m\times m}$ is nonsingular.

The control problem is then to design a controller using the Proportional-Integral sliding mode given by equation (8) such that the system state trajectory $X(t)$ tracks the desired state trajectory $X_d(t)$ as closely as possible for all t in spite of the uncertainties and non-linearities present in the system. The whole task can be divided into two parts; firstly it must be assured that the system error dynamics is asymptotically stable (approaching zero) during the sliding mode, and secondly a sliding mode controller is designed in such a way that whatever error the system has during the initial stage, the system must be directed towards the sliding surface (without sacrificing the stability aspect of the controlled system) during the reaching phase.

3 System Dynamics During Sliding Mode

Differentiating equation (8) gives

$$\dot{\sigma}(t) = C\dot{Z}(t) - [CA + CBK]Z(t) \quad (10)$$

Substituting equation (7) into equation (10) gives:

$$\dot{\sigma}(t) = CBH(t)Z(t) + CBH(t)X_d(t) - CB\Omega(t) + [CB + CBE(t)]u(t) - CBKZ(t) \quad (11)$$

Equating equation (11) to zero gives the equivalent control, $u_{eq}(t)$:

$$u_{eq}(t) = [CB + CBE(t)]^{-1}\{CBKZ(t) + CB\Omega(t) - CBH(t)Z(t) - CBH(t)X_d(t) \quad (12)$$

Noting that

$$[CB + CBE(t)]^{-1} = [I_n + E(t)]^{-1}(CB)^{-1} \quad (13)$$

the equivalent control of equation (12) can be written as

$$u_{eq}(t) = -[I_n + E(t)]^{-1}\{(H(t) - K)Z(t) - \Omega(t) + H(t)X_d(t)\} \quad (14)$$

Remark 1: The equivalent control $u_{eq}(t)$ is only a mathematically derived tool for the analysis of a sliding motion rather than a real control law generated in practical systems. In fact it is not realizable in the real controller.

The system dynamics during sliding mode can be found by substituting the equivalent control of equation (14) into the system error dynamics of equation (7):

$$\begin{aligned}\dot{Z}(t) &= [A + BH(t)]Z(t) + BH(t)X_d(t) - B\Omega(t) \\ &\quad -[B + BE(t)][I_n + E(t)]^{-1}\{[H(t) - K]Z(t) \\ &\quad - \Omega(t) + H(t)X_d(t)\} \\ &= [A + BK]Z(t) \end{aligned} \quad (15)$$

Hence if the matching condition is satisfied (equation (5) holds), the system's error dynamics during sliding mode is independent of the system uncertainties and couplings between the inputs, and, insensitive to the parameter variations.

4 Sliding Mode Tracking Controller Design

The manifold of equation (8) is asymptotically stable in the large, if the following hitting condition is held [5]:

$$(\sigma^T(t)/\|\sigma(t)\|)\dot{\sigma}(t) < 0 \quad (16)$$

As a proof, let the positive definite function be

$$V(t) = \|\sigma(t)\| \quad (17)$$

Differentiating equation (17) with respect to time, t yields

$$\dot{V}(t) = (\sigma^T(t)\dot{\sigma}_i(t))/\|\sigma(t)\| \quad (18)$$

Following the Lyapunov stability theory, if equation (16) holds, then the sliding manifold $\sigma(t)$ is asymptotically stable in the large.

Theorem: The hitting condition (16) of the manifold given by equation (8) is satisfied if the control $u(t)$ of system (7) is given by :

$$u(t) = -(CB)^{-1}[\gamma_1\|Z(t)\| + \gamma_2\|X_d(t)\| + \gamma_3\|\Omega(t)\|]SGN(\sigma(t)) + \Omega(t) \quad (19)$$

where

$$\gamma_1 > (\alpha\|CB\| + \|CBK\|)/(1+\beta) \quad (20)$$

$$\gamma_2 > (\alpha\|CB\|)/(1+\beta) \quad (21)$$

$$\gamma_3 > (\beta\|CB\|)/(1+\beta) \quad (22)$$

Proof: Substituting equation (19) into equation (11) gives:

$$\begin{aligned}\dot{\sigma}(t) &= CB[H(t) - K]Z(t) - (CB)[I_n + E(t)](CB)^{-1}\{\gamma_1\|Z(t)\| \\ &\quad + \gamma_2\|X_d(t)\| + \gamma_3\|\Omega(t)\|\}SGN(\sigma(t)) + CBH(t)X_d(t) \\ &\quad + CBE(t)\Omega(t) \end{aligned} \quad (23)$$

Substituting equation (23) into equation (18) gives the rate of change of the Lyapunov function:

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \quad (24)$$

where

$$\dot{V}_1(t) = (\sigma^T(t)/\|\sigma(t)\|)\{CB[H(t) - K]Z(t) - (CB)[I_n + E(t)](CB)^{-1}\gamma_1\|Z(t)\|SGN(\sigma(t))\} \quad (25)$$

$$\dot{V}_2(t) = (\sigma^T(t)/\|\sigma(t)\|)\{CBH(t)X_d(t) - (CB)[I_n + E(t)](CB)^{-1}\gamma_2\|X_d(t)\|SGN(\sigma(t))\} \quad (26)$$

$$\dot{V}_3(t) = (\sigma^T(t)/\|\sigma(t)\|)\{CBE(t)\Omega(t) - (CB)[I_n + E(t)](CB)^{-1}\gamma_3\|\Omega(t)\|SGN(\sigma(t))\} \quad (27)$$

Now let simplify each Lyapunov term individually. First term of equation (25):

$$
\begin{aligned}
&(\sigma^T(t)/\|\sigma(t)\|)\{CB[H(t)-K]\}Z(t) \\
&\quad \le (\|\sigma^T(t)\|/\|\sigma(t)\|)\{\|CB\|\|H(t)\|+\|CBK\|\}\|Z(t)\| \\
&\quad = \{\alpha\|CB\|+\|CBK\|\}\|Z(t)\|
\end{aligned}
\tag{28}
$$

Noting that

$$(\sigma^T(t)/\|\sigma(t)\|)SGN(\sigma(t)) = 1 \tag{29}$$

then the second term of equation (25) can be simplified as

$$
\begin{aligned}
&-\frac{\sigma^T(t)}{\|\sigma(t)\|}\{(CB)[I_n+E(t)](CB)^{-1}\gamma_1\|Z(t)\|SGN(\sigma(t))\} \\
&\qquad \le -\|CB\|[\|I_n\|+\|E(t)\|]\|(CB)^{-1}\|\gamma_1\|Z(t)\| \\
&\qquad = -(1+\beta)\gamma_1\|Z(t)\|
\end{aligned}
\tag{30}
$$

Using equations (28) and (30), equation (25) can be written as

$$\dot{V}_1(t) \le -\{(1+\beta)\gamma_1 - [\alpha\|CB\|+\|CBK\|]\}\|Z(t)\| \tag{31}$$

Similarly, equations (26) and (27) can be simplified in a same manner and the results are summarized as follows:

$$\dot{V}_2(t) \le -\{(1+\beta)\gamma_2 - \alpha\|CB\|\}\|X_d(t)\| \tag{32}$$

$$\dot{V}_3(t) \le -\{(1+\beta)\gamma_3 - \beta\|CB\|\}\|\Omega(t)\| \tag{33}$$

Let equations (20), (21) and (22) hold, then the global hitting condition (16) is satisfied. ∪

Remark 2: The conditions imposed by equations (20), (21) and (22) not only guarantee that the hitting condition (16) is met, but it also assure that based on the Lyapunov theory, the system dynamics is stable in the large.

5 Simulation Example

Consider a two-link manipulator (in the horizontal plane) with rigid links of nominally equal length l and mass m shown in Figure 1. The dynamics of the manipulator is [7]:

$$
\begin{aligned}
\ddot{\theta}_1 &= \frac{(\frac{2}{3}+\cos\theta_2)\sin\theta_2.\dot{\theta}_1+\frac{2}{3}\sin\theta_2.(2\dot{\theta}_1+\dot{\theta}_2).\dot{\theta}_2}{\frac{16}{9}-\cos^2\theta_2} \\
&\quad +\frac{\frac{4}{3}T_1-2(\frac{2}{3}+\cos\theta_2)T_2}{\frac{16}{9}-\cos^2\theta_2}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\theta}_2 &= \frac{-2(\frac{5}{3}+\cos\theta_2)\sin\theta_2.\dot{\theta}_1}{\frac{16}{9}-\cos^2\theta_2} \\
&\quad -\frac{(\frac{2}{3}+\cos\theta_2)\sin\theta_2.(2\dot{\theta}_1+\dot{\theta}_2).\dot{\theta}_2}{\frac{16}{9}-\cos^2\theta_2} \\
&\quad -\frac{2(\frac{2}{3}+\cos\theta_2)T_1-4(\frac{5}{3}+\cos\theta_2)T_2}{\frac{16}{9}-\cos^2\theta_2}
\end{aligned}
$$

Define

$$X(t)\underline{\underline{\Delta}}[x_1 \quad x_2 \quad x_3 \quad x_4]^T = \begin{bmatrix}\theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2\end{bmatrix}^T$$

$$U(t)\underline{\underline{\Delta}}[u_1 \quad u_2]^T = [T_1 \quad T_2]^T$$

Then the plant can be represented in the form of

$$\dot{X}(t) = A(t)X(t) + B(t)u(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & 0 & 0_{44} \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \\ 0 & 0 \\ b_{41} & b_{42} \end{bmatrix}$$

$$a_{22} = ((2/3)+\cos x_3)\sin x_3 \cdot x_2/((16/9)-\cos^2 x_3)$$
$$b_{24} = (2/3)\sin x_3 \cdot (2x_2+x_4)\cdot x_4/((16/9)-\cos^2 x_3)$$
$$a_{42} = -2((5/3)+\cos x_3)\sin x_3 \cdot x_2/((16/9)-\cos^2 x_3)$$
$$a_{44} = -((2/3)+\cos x_3)\sin x_3(2x_2+x_4)/((16/9)-\cos^2 x_3)$$
$$b_{21} = (4/3)/((16/9)-\cos^2 x_3)$$
$$b_{22} = -2((2/3)+\cos x_3)/((16/9)-\cos^2 x_3)$$
$$b_{42} = b_{22}$$
$$b_{42} = 4((5/3)+\cos x_3)/((16/9)-\cos^2 x_3)$$

Suppose that the bounds of the $\theta_i(t)$ and $\dot{\theta}_i(t)$ are:

$$-150^\circ \le \theta_1 \le 150^\circ,\ 0^\circ s^{-1} \le \dot{\theta}_1 \le 50^\circ s^{-1},$$

$$-35^\circ \le \theta_2 \le 100^\circ,\ 0^\circ s^{-1} \le \dot{\theta}_2 \le 30^\circ s^{-1}$$

With these bounds, the plant can be represented in the form of equation (1) with the nominal value of A and B calculated as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1.1684 & 0 & 0.9724 \\ 0 & 0 & 0 & 1 \\ 0 & -2.6496 & 0 & -2.4310 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1.5058 & -1.9489 \\ 0 & 0 \\ -1.9489 & 8.4151 \end{bmatrix}$$

The uncertainties for system and input matrices are

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.9744 & 0 & 0.9724 \\ 0 & 0 & 0 & 0 \\ 0 & 4.2076 & 0 & 2.4310 \end{bmatrix}$$

$$\Delta B = \begin{bmatrix} 0 & 0 \\ 0.2085 & 0.3368 \\ 0 & 0 \\ 2.3368 & 5.2991 \end{bmatrix}$$

Using equation (5), the bounds of $H(t)$ and $E(t)$ can be computed:

$$\|H(t)\| \le \alpha = 2.6046\ ;\ \|E(t)\| \le \beta = 1.9617$$

It is assumed that each sub-system is required to track a pre-specified cycloidal function of the form:

$$\theta_{di}(t) = \begin{cases} \theta_i(0) + \frac{\Delta_i}{2\pi}[\frac{2\pi t}{\tau} - \sin(\frac{2\pi t}{\tau})], & 0 \le t \le \tau \\ \theta_i(\tau), & \tau \le t \end{cases}$$

where $\Delta_i = \theta_i(\tau) - \theta_i(0)$, $i = 1,2$. In this example, the input trajectory data used are as follows:

Start time, $t(0) = 0.0\ s$

Final time, $\tau = 10.0\ s$

Start positions, $\theta_1(0) = 10$ deg ; $\theta_2(0) = 15$ deg

Final positions, $\theta_1(\tau) = 50$ deg ; $\theta_2(\tau) = 60$ deg

Define the gains:

$$K = \begin{bmatrix} 2.0125 & 3.4291 & 0.0919 & 0.8735 \\ 0.3235 & 0.4080 & 0.6868 & 0.4838 \end{bmatrix}$$ such that

$$\lambda(A + BK) = \{-1, -2, -2, -3\}$$

and $$C = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 1 \end{bmatrix}$$

Remark 3: The gain K can be chosen arbitrarily but in this paper the values of K is intentionally selected to represent the case of over-damped response. Since the choice of C affects the system response during the reaching phase, it must be appropriately selected to give the desired result. Nevertheless, C must be chosen such that CB is nonsingular.

The controller parameter γ's can therefore be computed from equation (20)-(22):

$$\gamma_1 > 1.1731;\ \gamma_2 > 0.8742;\ \gamma_3 > 0.6584$$

For comparison purposes, two sets of the controller parameters are chosen:

Case1: $\gamma_1 = 0.5;\ \gamma_2 = 0.2;\ \gamma_3 = 0.2$

Case2: $\gamma_1 = 1.5;\ \gamma_2 = 1.0;\ \gamma_3 = 1.0$

In Case 1, the controller parameter is selected to study the performance of the system if the gain conditions of equations (20)-(22) are not met; while in Case 2 the controller parameters is selected to represent a situation where the conditions imposed on the controller are met. The simulation results for Case 1 and Case 2 are shown in Figure 2 and Figure 3, respectively. If the controller parameter conditions are not met (Case 1), the actual output positions fail to track the desired positions (Figure 2a and Figure 2b). This is due to the fact that the control inputs not succeed to switch fast enough (Figure 2c and Figure 2d) and hence the sliding mode fails to materialized (Figure 2e).

On the contrary, when the controller parameter conditions are met (Case 2), the position tracking is satisfactory (Figure 3a and Figure 3b). As expected, the control inputs switch indiscriminately very fast (Figure 3c and Figure 3d), resulting the sliding surfaces to converge to zero (Figure 3e) and hence making the sliding phase took place.

6 Conclusions

In this paper, a Proportional-Integral Sliding Mode controller is proposed for a class of uncertain system. It is shown mathematically that the error dynamics during sliding mode is stable and can easily be shaped-up using the conventional pole-placement technique. Beside during the sliding phase, the system stability is also guaranteed during the reaching phase. Application to a two-link planar robot manipulator is presented to illustrate the effectiveness of the proposed controller.

References

[1]. De Carlo, R. A., Zak, S. H., and Matthews, G. P., Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial, *Proc. IEEE*, Vol. 76, No. 3, 1988, pp. 212-232.

[2]. Hung, J.Y., Gao, W, and Hung, J.C., Variable Structure Control: A Survey, *Trans. IEEE Industrial Electronics*, Vol. 40, No. 1, 1993, pp. 2-22.

[3]. Young, K. D., Utkin, V. I., and Ozguner, U., A Control Engineer's Guide to Sliding Mode Control, *Proc. IEEE Int. Workshop on Variable Structure Systems (VSS 96)*, 1996, pp. 1-14.

[4]. Utkin, V., and Shi, J., Integral Sliding Mode in Systems Operating under Uncertainty Conditions, *Proc. IEEE Conf. Decision and Control*, Kobe, Japan, 1996, pp. 4591-4596.

[5]. Yan, J. -J, Tsai, J. S-H, Kung, F.-C., Robust Decentralized Stabilization of Large-Scale Delay Systems Via Sliding Mode Control, *Trans. ASME*, Vol. 119, 1997, pp. 307-312.

[6]. Wu, H., Robust Tracking and Model Following for a Class of Uncertain Dynamic Systems by Variable Structure Control, *Proc. IEEE Int. Conf. Control Applications*, Alaska, USA, Sept. 25-27, 2000, pp. 680-685.

[7]. Slotine, J. J., and Sastry, S. S., Tracking Control of Nonlinear Systems using Sliding Surfaces, with Application to Robot Manipulators, *Int. J. Control*, Vol. 38, No. 2, 1983, pp. 465-492.

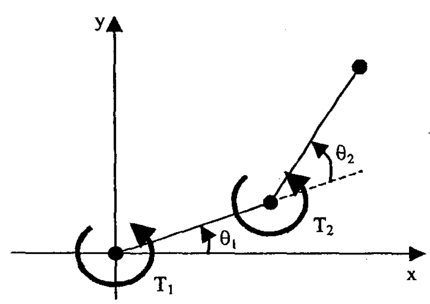
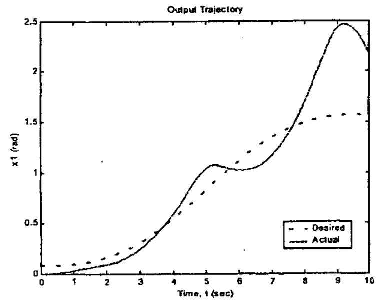
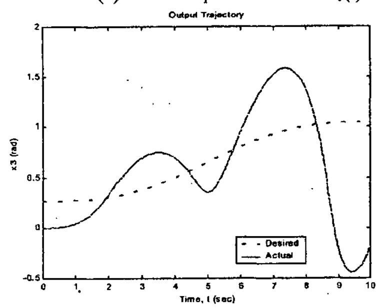


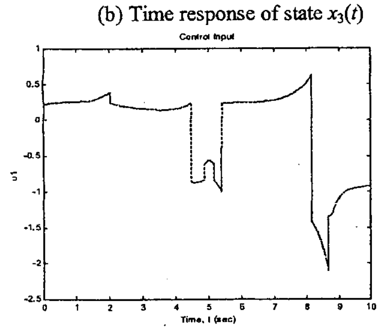
Figure 1: A two-link manipulator.



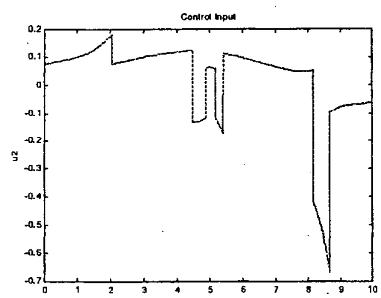
(a) Time response of state $x_1(t)$



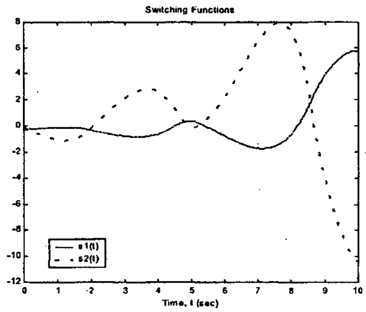
(b) Time response of state $x_3(t)$



(c) Control input $u_1(t)$

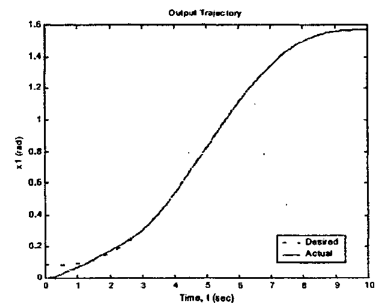


(d) Control input $u_2(t)$

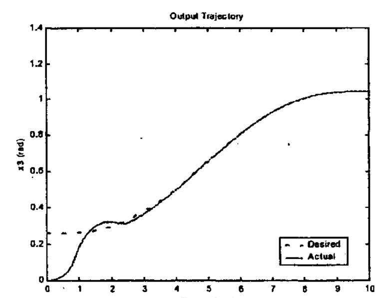


(e) Switching functions $\sigma_1(t)$ and $\sigma_2(t)$

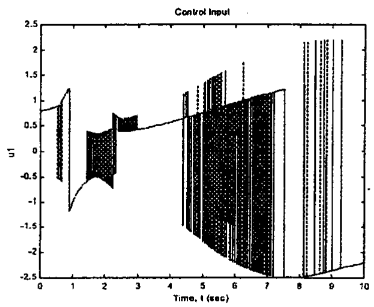
Figure 2: Simulation results for Case 1.



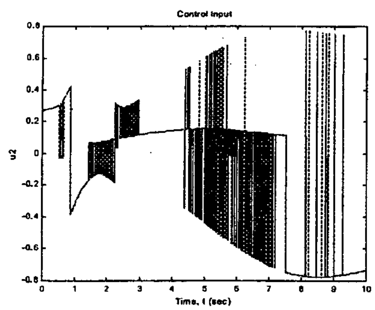
(a) Time response of state $x_1(t)$



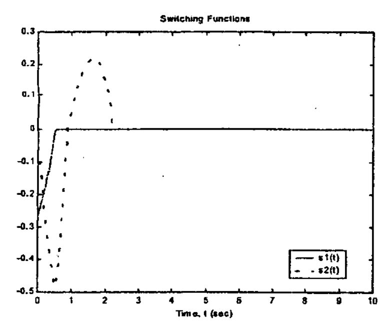
(b) Time response of state $x_3(t)$



(c) Control input $u_1(t)$



(d) Control input $u_2(t)$



(e) Switching functions $\sigma_1(t)$ and $\sigma_2(t)$

Figure 3: Simulation results for Case 2.