

THREE-PARAMETER LOGNORMAL DISTRIBUTION: PARAMETRIC ESTIMATION USING L-MOMENT AND TL-MOMENT APPROACH

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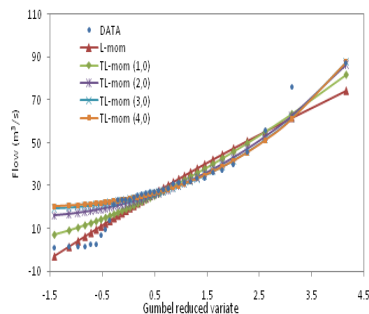
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Graphical abstract



Abstract

The three-parameter lognormal (LN3) distribution has been applied to the frequency analysis of flood events. L-moment and TL-moment methods are applied in estimating parameters of the LN3 distribution which are L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 to the LN3 distribution. A simulation study is conducted in this paper by fitting this distribution to generate LN3 and non LN3 samples. Relative Root Mean Square Error (RRMSE) and relative bias are evaluated to illustrate the performance of this distribution. The performance of TL-moments approach was compared with L-moments based on the streamflow data from Sg. Trolak and Sg. Slim which are located in Perak, Malaysia. The results showed that TL-moments approach produced a better result at high quantile estimation compared to L-moments.

Keywords: L-moments, TL-moments, three-parameter lognormal distribution

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1.0 INTRODUCTION

The three-parameter lognormal distribution (LN3) is one of the most versatile distributions. Its application can be seen in various fields such as agriculture, entomology, economics, geology, industry, quality control and hydrology. In finance, Levy and Kroll [1] used the three-parameter lognormal distribution (LN3) to derive an efficient investment rule. The lognormal distribution is also useful in modeling data which would be considered normally distributed except for the fact that it may be more or less skewed. Such skewness occurs frequently when means are low, variances are large and values cannot be negative [2].

In literature, the studies on lognormal distribution are relatively scarce. Some studies applied this distribution on the L-moment method with historical flood [3–7] and wage [8–10]. From these studies, L-moments method was shown as the superior method for

parameter estimation method of LN3 distribution whether in wage and income area and historical flood. L-moments sometimes bring even more efficient parameter estimations of the parametric distribution than those estimated by the maximum likelihood method for small samples in particular [11].

Bilkova [8] had done the application of L-moments in the case of larger samples for wage and the comparison of the precision of L-moments with other methods (moment, quantile and maximum likelihood method) of parameter estimation the case of larger samples. She used two types of data, namely data sets of individual data and data ordered to the form of interval frequency distribution and the LN3 distribution was used as the theoretical model. From her studies, the method of L-moments provides the most accurate results, which are even more accurate than the results obtained using the maximum likelihood method.

Other researchers applied the parameter estimation of LN3 distribution on historical flood [7]. They compared the ordinary moment method; the curve-fitting method with absolute norm and the L-moment estimation were also carried out using Monte-Carlo experiments. The results showed that L-moment estimation is the preferred method and therefore recommended to be used in their practice.

Similarly, this study will use L-moment method as estimation parameter method of LN3 distribution. Since severe literatures proved that L-moment method produces a better result than maximum likelihood method [8], moment method [7-9], this paper will choose Trimmed L-moments (TL-moments) method to compare with L-moment method in estimating parameter of LN3 distribution.

TL-moment was introduced by Elamir and Seheult [12] with a view to increase the awareness towards the outliers. The TL-moments give zero weight to the extreme value, are easy to compute and are said to be more robust than the L-moments in the presence of outliers. A few studies have been done regarding on TL-moments method for several distributions.¹³⁻²¹

Bilkova [22-25] presented her papers that deal with an alternative approach to the construction of an appropriate parametric distribution for the considered data set using order statistics L-moment and TL-moment method of generalized Pareto (GPA) and LN3 distribution. The research variable is the net annual household income per capita (in CZK) in the Czech Republic (nominal income). As a conclusion, she found out that some theoretical and practical aspects of TL-moments are still the subject of research or they are remain for future research.

However, there is no further research investigating on flood frequency analysis on estimation parameter of LN3 distribution by L-moment and TL-moment method. This paper will deal to derive the expressions of TL-moments from the annual streamflows data and to fit the parameters of the LN3 distribution using TL-moments approach. Monte Carlo simulation study is conducted to compare the performance of L-moment and TL-moment approach for the LN3 distribution. More specifically, we develop the method of TL-moment for the LN3 distribution, which is often employed in statistical analyses of hydrological data. The performance of the proposed estimators of the LN3 distribution was compared with the estimators based on L-moment estimators for various sample sizes and return periods.

2.0 THREE-PARAMETER LOG-NORMAL (LN3) DISTRIBUTION

The lognormal distribution finds its beginning in 1879 by F. Galton. The probability density function of three-parameter log-normal (LN3) distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi}(x-\alpha)\sigma} \exp\left(-\frac{1}{2}\left[\frac{\ln(x-\alpha)-\mu}{\sigma}\right]^2\right) \quad (1)$$

where $y = \ln(x - \alpha)$ has a normal distribution with mean, μ and standard deviation, σ of the random variable's logarithm while α is scale or lower bound parameter. Additionally, μ is said to be a location parameter and σ is said to be a shape parameter of the lognormal density function. The corresponding cumulative distribution function is:

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \quad (2)$$

Hosking [26] used a modification of the Munro and Wixley [27] parameterization, $\kappa = -\sigma$, $\alpha = \sigma e^\mu$, and $\xi = \gamma + e^\mu$. In this case, the LN3 distribution is also called the Generalized Normal distribution. It is often more advantageous to use a parameterization of the lognormal distribution. The advantage of this parameterization is that the LN3 distribution can fit data with positive skewness and a lower limit and negative skewness and an upper limit, while it also includes the normal distribution as a special case [28].

3.0 METHOD OF PARAMETER ESTIMATION

3.1 L-Moment Method

Let $x_1 \leq x_2 \leq \dots \leq x_n$ be the ordered sample of size n drawn from the distribution of x . The corresponding order statistics of sample size can be denoted by, $x_{1:n} \leq x_{2:n} \leq \dots \leq x_{n:n}$. Hosking [11] defines the r^{th} L-moments of x as

$$\lambda_r \equiv \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:k}), \quad r = 0, 1, \dots, n \quad (3)$$

The expected value of order statistics defined by Elamir and Seheult [12] is

$$E(Y_{n,r}) = r \binom{n}{r} \int_0^1 Q(u) u^{r-1} (1-u)^{n-r} du, \quad r \leq n \quad (4)$$

where $\beta_r = \int_0^1 Q(u) u^r du$. From Eqs. (3)–(4), the first four L-moments can be derived as follow:

$$\lambda_1 = \beta_0, \quad (5)$$

$$\lambda_2 = 2\beta_1 - \beta_0, \quad (6)$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0, \quad (7)$$

$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \quad (8)$$

L-moment ratios are defined by Hosking [11] as:

$$\text{L-coefficient of variation (L-CV)}, \quad \tau_2 = \frac{\lambda_2}{\lambda_1} \quad (9)$$

$$\text{L-coefficient of skewness (L-CS)}, \quad \tau_3 = \frac{\lambda_3}{\lambda_2} \quad (10)$$

$$\text{L-coefficient of kurtosis (L-CK)}, \quad \tau_4 = \frac{\lambda_4}{\lambda_2} \quad (11)$$

3.2 TL-Moment Method

Elamir and Seheult [12] defined the r^{th} TL-moment as

$$\lambda_r^{(t_1, t_2)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+t_1-k:r+t_1+t_2}) \quad (12)$$

where $r = 1, 2, \dots$ and the Eq. (4) can be used as the expected value of order statistics and $t_1 = 1, 2, 3$, and 4 while $t_2 = 0$. Since $t_2 = 0$, $\lambda_r^{(t_1, t_2)}$ can be denoted as $\lambda_r^{(\eta, 0)} = \lambda_r^\eta$ where $\eta = 0, 1, 2, 3$, and 4. The sample TL-moment was presented by Elamir and Seheult [12] is:

$$l_r^{(t_1, t_2)} = \frac{1}{r \binom{n}{r+t_1+t_2}} \sum_{i=t_1+1}^{n-t_2} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-t_1-k-1} \binom{n-i}{k+t_2} x_{i:n} \quad (13)$$

where $t_1 = 1, 2, 3$, and 4 while $t_2 = 0$, r represents the order of the L-moment, n represents the sample size, and $x_{i:n}$ is the i th sample order statistic. Since $t_2 = 0$, $l_r^{(t_1, t_2)}$ can be denoted as $l_r^{(\eta, 0)} = l_r^\eta$ where $\eta = 0, 1, 2, 3$, and 4.

From Eq. (12), the first four TL-moments $\eta = 1$ can be derived as follow:

$$\lambda_1^1 = 2\beta_1, \quad (14)$$

$$\lambda_2^1 = \frac{1}{2}(9\beta_2 - \beta_1), \quad (15)$$

$$\lambda_3^1 = \frac{1}{3}(40\beta_3 - 48\beta_2 + 12\beta_1), \quad (16)$$

$$\lambda_4^1 = \frac{1}{4}(175\beta_4 - 300\beta_3 + 150\beta_2 - 20\beta_1) \quad (17)$$

TL-moments $\eta = 1$ ratio can be defined as:

$$\text{TL-coefficient of variation (TL-CV), } \tau_2^1 = \frac{\lambda_2^1}{\lambda_1^1} \quad (18)$$

$$\text{TL-coefficient of skewness (TL-CS), } \tau_3^1 = \frac{\lambda_3^1}{\lambda_2^1} \quad (19)$$

$$\sigma = a_0 + a_1\tau_3^\eta + a_2(\tau_3^\eta)^2 + a_3(\tau_3^\eta)^3 + a_4(\tau_3^\eta)^4 + a_5(\tau_3^\eta)^5 + a_6(\tau_3^\eta)^6 + a_7(\tau_3^\eta)^7 + a_8(\tau_3^\eta)^8 \quad (25)$$

where the coefficients, a vary with η and are given in the Table 2.

Table 1 Parameter estimates for L-moment and TL-moment of the LN3 distribution

Parameter Estimates		
η	Location, α	Scalar, μ
0	$\alpha = l_1 - e^{\mu + \frac{\sigma^2}{2}}$	$\mu = \ln(l_2) - \ln(2S_1(\sigma) - e^{\frac{\sigma^2}{2}})$
1	$\alpha = l_1 - 2S_1(\sigma)e^\mu$	$\mu = \ln(2l_2) - \ln(9S_2(\sigma) - 6S_1(\sigma))$
2	$\alpha = l_1 - 3S_2(\sigma)e^\mu$	$\mu = \ln(l_2) - \ln(8S_3(\sigma) - 6S_2(\sigma))$

$$\text{TL-coefficient of kurtosis (TL-CK), } \tau_4^1 = \frac{\lambda_4^1}{\lambda_2^1} \quad (20)$$

3.3 L-Moments and TL-Moments of the LN3 Distribution

The L-moments and TL-moments of the LN3 are obtained as follow:

$$\beta_r = \frac{\alpha}{r+1} + S_r(\sigma) \exp(\mu), \quad r = 0, 1, 2, \dots \quad (21)$$

where

$$S_r(\sigma) = \int_0^1 F^r(x) \exp(x\sigma) dF(x) = \int_0^1 \exp(x\sigma) \cdot \left(\int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \right)^r \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (22)$$

$r = 0, 1, 2, \dots$

For TL-moments $\eta = 2, 3$, and 4 for LN3 distribution, the steps to find the first four TL-moments and TL-moments ratios are the same as the TL-moments $\eta = 1$ which used Eq. (12) and Eqs. (18)–(20). The three parameters α , μ , and σ in the LN3 distribution can be estimated by matching the first three TL-moments to their sample estimates for a selected η . The scale and location parameter for each level of TL-moments estimated in Eqs. (23)–(24).

$$\alpha = (\eta + 1)(\beta_\eta - e^\mu S_\eta) \quad (23)$$

$$\mu = \ln\left(\frac{(\eta + 2)\beta_{\eta+1} - (\eta + 1)\beta_\eta}{(\eta + 2)S_{\eta+1} - (\eta + 1)S_\eta}\right) \quad (24)$$

where $S_\eta = S_r(\sigma)$ $r = 0, 1, 2, \dots$. Given σ or Cs, the exact values of $S_r(\sigma)$ cannot be computed because it is impossible, analytically, to integrate the right-hand sides of Eq. (22). The scale and location parameter for L-moment and each level of TL-moments estimated as in Table 1. By using the relationships of TL-Cs of the LN3 distribution, the shape parameter was developed as Eq. (25).

Parameter Estimates		
η	Location, α	Scalar, μ
3	$\alpha = l_1 - 4S_3(\sigma)e^{\mu}$	$\mu = \ln(2l_2) - \ln(25S_4(\sigma) - 20S_3(\sigma))$
4	$\alpha = l_1 - 5S_4(\sigma)e^{\mu}$	$\mu = \ln(l_2) - \ln(18S_5(\sigma) - 15S_4(\sigma))$

Table 2 Values of coefficients of the shape parameter

η	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
0	-0.0010	0.9974	0.0010	-0.0059	-0.0002	0.0001	0	0	0
1	-0.2938	1.2503	0.0816	-0.0027	-0.0159	-0.0077	0.0025	0.0015	0.0002
2	-0.4713	1.4351	0.0689	-0.0281	0.0111	0.0052	0.0008	0.0001	0
3	-0.6251	1.6352	0.1671	-0.0971	-0.0507	0.0113	0.0151	0.0038	0.0003
4	-0.7520	1.8172	0.2770	-0.1782	-0.1316	0.0177	0.0340	0.0089	0.0007

4.0 MONTE CARLO SIMULATION

Monte Carlo simulation study is conducted to compare the performance L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 for the LN3 distribution. The measures of performance that were used are the Relative Bias (RBIAS) and the Relative Root Mean Square Error (RRMSE).

The RRMSE and RBIAS can be represented as

$$RRMSE = \sqrt{\frac{1}{M} \sum_{m=1}^M \left(\frac{\hat{x}_i - x}{\hat{x}_i} \right)^2} \tag{26}$$

$$RBIAS = \frac{1}{M} \sum_{m=1}^M \left(\frac{\hat{x}_i - x}{\hat{x}_i} \right) \tag{27}$$

where M is the sample size, \hat{x}_i and x is the estimated values and true value of the quantile. In each simulation, 5 000 samples are used for sample size 15, 30, 50, and 100. The quantile function $x(F)$ that has been obtained for RRMSE and RBIAS values are $F = 0.9, F = 0.95, F = 0.98, F = 0.99,$ and $F = 0.995,$ i.e; $Q_{10}, Q_{20}, Q_{50}, Q_{100},$ and $Q_{200}.$

4.1 Known Parent Distribution

The known parent distribution is useful to see the impact on the estimation when the assumed distribution function is similar to the parent distribution function. In this study, the LN3 distribution function is fitted to the generated LN3 samples. In this simulation, the value of parameter of location and scale is set to 0 and 1 [29] with shape parameter k is between 0.2 and 2.0.

Tables 3 and 4 present the RRMSE and RBIAS values of quantiles $x(F)$ for $F = 0.9, F = 0.95, F = 0.98, F = 0.99,$ and $F = 0.995$ computed using L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 for the LN3 distribution for all samples, $n = 15, 30, 50,$ and $100.$ Table 3 shows the results for small sample size, $n = 15$ where the TL-moment $\eta = 4$ produce the smaller value of RRMSE at $1.0 \leq \sigma \leq 2.0$ for quantile $x(F = 0.995)$ compare to L-moment. However, at $0.2 \leq \sigma \leq 0.8$ the larger RRMSE value was given by TL-moments $\eta = 4.$

For the result of RBIAS values, the results from Table 4 show that the smaller $\sigma,$ the quantile estimates for L-moment and TL-moments are become more unbiased for mostly quantiles. When σ becomes larger, the quantile estimator becomes more positively biased.

Table 3 RRMSE of $x(F)$ estimator fitting the LN3 distribution to generated LN3 samples

σ	Method	n=15			n=30			n=50			n=100		
		F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995
0.2	0	0.072	0.091	0.112	0.050	0.061	0.074	0.038	0.047	0.056	0.027	0.032	0.038
	1	0.076	0.099	0.127	0.052	0.066	0.083	0.040	0.050	0.062	0.028	0.034	0.042
	2	0.076	0.103	0.137	0.053	0.069	0.088	0.040	0.052	0.065	0.028	0.035	0.044
	3	0.077	0.106	0.146	0.053	0.071	0.093	0.041	0.054	0.069	0.028	0.037	0.047
	4	0.076	0.107	0.151	0.053	0.072	0.097	0.041	0.055	0.072	0.029	0.038	0.049
0.4	0	0.164	0.208	0.258	0.114	0.142	0.171	0.088	0.108	0.129	0.061	0.074	0.088
	1	0.170	0.225	0.293	0.119	0.154	0.194	0.092	0.117	0.144	0.064	0.080	0.098
	2	0.170	0.231	0.309	0.120	0.158	0.203	0.093	0.120	0.151	0.064	0.082	0.102
	3	0.170	0.234	0.322	0.121	0.162	0.213	0.093	0.123	0.159	0.065	0.085	0.108
	4	0.168	0.235	0.328	0.120	0.164	0.219	0.093	0.125	0.164	0.065	0.087	0.112
0.6	0	0.272	0.349	0.437	0.193	0.241	0.293	0.148	0.183	0.219	0.103	0.125	0.149
	1	0.277	0.369	0.484	0.199	0.259	0.328	0.154	0.197	0.245	0.107	0.135	0.166
	2	0.274	0.372	0.502	0.199	0.263	0.341	0.154	0.200	0.254	0.108	0.138	0.172

	3	0.272	0.372	0.510	0.198	0.266	0.350	0.154	0.204	0.263	0.108	0.141	0.179
	4	0.270	0.371	0.513	0.197	0.266	0.355	0.154	0.205	0.268	0.108	0.143	0.184
0.8	0	0.389	0.504	0.640	0.282	0.355	0.438	0.217	0.270	0.325	0.151	0.185	0.220
	1	0.388	0.517	0.682	0.285	0.372	0.476	0.222	0.286	0.357	0.156	0.198	0.243
	2	0.381	0.514	0.693	0.282	0.375	0.489	0.221	0.288	0.367	0.156	0.201	0.250
	3	0.377	0.508	0.691	0.280	0.374	0.493	0.221	0.290	0.374	0.156	0.203	0.256
	4	0.375	0.503	0.686	0.279	0.373	0.495	0.220	0.290	0.377	0.156	0.204	0.260
1.0	0	0.504	0.657	0.847	0.375	0.479	0.599	0.292	0.365	0.446	0.205	0.252	0.301
	1	0.495	0.657	0.868	0.372	0.488	0.628	0.293	0.377	0.475	0.209	0.265	0.326
	2	0.484	0.646	0.868	0.365	0.484	0.635	0.289	0.378	0.484	0.207	0.267	0.334
	3	0.477	0.630	0.848	0.361	0.478	0.630	0.288	0.376	0.485	0.206	0.267	0.337
	4	0.476	0.621	0.832	0.361	0.474	0.625	0.288	0.376	0.485	0.206	0.268	0.339
1.2	0	0.608	0.794	1.034	0.464	0.599	0.762	0.366	0.464	0.575	0.275	0.337	0.404
	1	0.591	0.776	1.025	0.453	0.593	0.770	0.361	0.466	0.591	0.260	0.331	0.410
	2	0.583	0.767	1.024	0.444	0.585	0.770	0.355	0.463	0.596	0.257	0.333	0.419
	3	0.571	0.736	0.975	0.437	0.569	0.748	0.353	0.457	0.589	0.257	0.332	0.419
	4	0.574	0.724	0.946	0.438	0.563	0.733	0.355	0.456	0.584	0.258	0.332	0.418
1.4	0	0.696	0.905	1.183	0.552	0.712	0.913	0.440	0.559	0.702	0.316	0.397	0.485
	1	0.676	0.873	1.145	0.524	0.682	0.889	0.424	0.547	0.697	0.311	0.397	0.495
	2	0.681	0.883	1.169	0.521	0.680	0.894	0.418	0.543	0.701	0.306	0.395	0.501
	3	0.661	0.828	1.076	0.507	0.647	0.842	0.413	0.528	0.678	0.304	0.391	0.495
	4	0.668	0.814	1.035	0.511	0.638	0.817	0.417	0.526	0.668	0.308	0.391	0.493
1.6	0	0.771	0.990	1.289	0.607	0.786	1.019	0.494	0.632	0.804	0.367	0.463	0.574
	1	0.754	0.953	1.236	0.587	0.753	0.978	0.480	0.614	0.784	0.358	0.456	0.572
	2	0.786	1.007	1.321	0.601	0.778	1.019	0.482	0.622	0.801	0.352	0.455	0.577
	3	0.750	0.913	1.164	0.573	0.714	0.917	0.468	0.589	0.751	0.347	0.443	0.561
	4	0.762	0.899	1.111	0.581	0.703	0.881	0.475	0.585	0.734	0.352	0.442	0.555
1.8	0	0.838	1.056	1.360	0.661	0.846	1.096	0.543	0.693	0.884	0.410	0.520	0.650
	1	0.831	1.029	1.316	0.644	0.813	1.047	0.529	0.669	0.853	0.398	0.506	0.636
	2	0.912	1.158	1.504	0.694	0.893	1.163	0.554	0.710	0.911	0.400	0.514	0.652
	3	0.841	1.001	1.253	0.639	0.778	0.985	0.522	0.644	0.813	0.385	0.485	0.613
	4	0.859	0.985	1.186	0.649	0.763	0.935	0.532	0.638	0.786	0.392	0.483	0.601
2.0	0	0.903	1.112	1.407	0.708	0.891	1.141	0.587	0.741	0.941	0.446	0.564	0.707
	1	0.914	1.112	1.406	0.702	0.871	1.114	0.578	0.721	0.913	0.433	0.545	0.685
	2	1.077	1.360	1.749	0.818	1.049	1.357	0.647	0.824	1.052	0.457	0.584	0.737
	3	0.941	1.100	1.359	0.708	0.846	1.060	0.579	0.702	0.876	0.421	0.523	0.656
	4	0.963	1.077	1.271	0.718	0.822	0.987	0.592	0.691	0.835	0.428	0.516	0.635

* Values in bold indicate that RRMSE is smaller than TL-moments

Table 4 RBias of $x(F)$ estimator fitting the LN3 distribution to generated LN3 samples

Method	n=15			n=30			n=50			n=100			
	F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995	F=0.98	F=0.99	F=0.995	
0.2	0	-0.004	-0.007	-0.012	-0.002	-0.003	-0.006	-0.001	-0.002	-0.003	-0.0006	-0.001	-0.002
	1	-0.002	-0.007	-0.014	-0.001	-0.003	-0.006	0.000	-0.001	-0.003	-0.0002	-0.001	-0.001
	2	-0.001	-0.007	-0.015	-0.001	-0.003	-0.007	0.000	-0.002	-0.004	-0.0006	-0.001	-0.003
	3	0.001	-0.006	-0.018	-0.001	-0.004	-0.009	-0.001	-0.003	-0.007	-0.001	-0.003	-0.005
	4	0.003	-0.005	-0.018	0.000	-0.005	-0.011	-0.001	-0.004	-0.009	-0.001	-0.004	-0.007
0.4	0	-0.006	-0.015	-0.026	-0.003	-0.007	-0.012	-0.001	-0.004	-0.006	-0.001	-0.003	-0.004
	1	-0.003	-0.014	-0.031	-0.002	-0.008	-0.015	-0.001	-0.004	-0.009	-0.0014	-0.003	-0.005
	2	0.002	-0.011	-0.031	-0.001	-0.007	-0.015	0.000	-0.004	-0.009	-0.0011	-0.003	-0.006
	3	0.006	-0.009	-0.033	0.000	-0.008	-0.020	-0.001	-0.006	-0.014	-0.002	-0.006	-0.011
	4	0.011	-0.005	-0.032	0.002	-0.008	-0.022	0.000	-0.007	-0.018	-0.002	-0.008	-0.016
0.6	0	-0.007	-0.021	-0.041	-0.005	-0.012	-0.021	-0.002	-0.006	-0.012	-0.002	-0.004	-0.007
	1	0.000	-0.019	-0.046	-0.002	-0.013	-0.026	-0.002	-0.008	-0.017	-0.003	-0.007	-0.011
	2	0.009	-0.011	-0.042	0.002	-0.008	-0.023	0.001	-0.005	-0.014	-0.0010	-0.004	-0.009
	3	0.016	-0.005	-0.040	0.004	-0.008	-0.027	0.001	-0.007	-0.020	-0.002	-0.008	-0.016
	4	0.024	0.003	-0.035	0.007	-0.007	-0.028	0.003	-0.008	-0.023	-0.0008	-0.009	-0.021
0.8	0	-0.003	-0.025	-0.054	-0.005	-0.016	-0.031	-0.002	-0.009	-0.018	-0.003	-0.006	-0.011
	1	0.008	-0.017	-0.055	0.000	-0.015	-0.037	0.000	-0.011	-0.025	-0.003	-0.010	-0.018
	2	0.022	-0.003	-0.044	0.008	-0.006	-0.028	0.005	-0.004	-0.018	0.000	-0.005	-0.012
	3	0.033	0.009	-0.035	0.011	-0.004	-0.028	0.006	-0.005	-0.021	0.001	-0.007	-0.017

	4	0.043	0.021	-0.024	0.015	0.000	-0.026	0.008	-0.004	-0.022	0.002	-0.008	-0.021
1.0	0	0.007	-0.021	-0.060	-0.001	-0.017	-0.040	0.000	-0.010	-0.024	-0.002	-0.008	-0.015
	1	0.025	-0.006	-0.052	0.008	-0.011	-0.040	0.004	-0.010	-0.029	-0.001	-0.010	-0.022
	2	0.043	0.015	-0.033	0.018	0.001	-0.027	0.011	0.000	-0.018	0.004	-0.003	-0.013
	3	0.057	0.033	-0.016	0.024	0.008	-0.021	0.014	0.003	-0.016	0.005	-0.003	-0.014
	4	0.068	0.048	0.001	0.027	0.013	-0.015	0.016	0.005	-0.014	0.006	-0.002	-0.015
1.2	0	0.027	-0.006	-0.053	0.010	-0.012	-0.042	0.006	-0.008	-0.027	-0.001	-0.008	-0.019
	1	0.050	0.018	-0.034	0.022	0.000	-0.034	0.014	-0.003	-0.026	0.004	-0.007	-0.021
	2	0.070	0.042	-0.010	0.034	0.016	-0.017	0.023	0.010	-0.013	0.010	0.002	-0.010
	3	0.087	0.066	0.017	0.041	0.026	-0.004	0.026	0.015	-0.005	0.011	0.005	-0.007
	4	0.099	0.085	0.041	0.044	0.034	0.006	0.027	0.019	0.001	0.011	0.006	-0.004
1.4	0	0.057	0.022	-0.030	0.026	0.001	-0.035	0.018	0.001	-0.024	0.007	-0.003	-0.016
	1	0.082	0.053	0.000	0.043	0.020	-0.017	0.029	0.012	-0.014	0.013	0.002	-0.014
	2	0.100	0.075	0.022	0.055	0.037	0.002	0.038	0.025	0.001	0.019	0.011	-0.003
	3	0.122	0.107	0.062	0.064	0.053	0.023	0.042	0.035	0.014	0.020	0.016	0.005
	4	0.135	0.130	0.091	0.066	0.062	0.038	0.041	0.039	0.023	0.018	0.018	0.010
1.6	0	0.095	0.062	0.009	0.054	0.029	-0.011	0.038	0.020	-0.008	0.019	0.008	-0.009
	1	0.121	0.096	0.047	0.070	0.051	0.014	0.049	0.035	0.008	0.026	0.016	-0.001
	2	0.130	0.110	0.061	0.077	0.062	0.027	0.056	0.046	0.021	0.033	0.026	0.010
	3	0.159	0.154	0.114	0.090	0.086	0.060	0.062	0.060	0.042	0.033	0.033	0.023
	4	0.173	0.178	0.149	0.092	0.097	0.079	0.060	0.065	0.054	0.028	0.034	0.030
1.8	0	0.139	0.112	0.062	0.087	0.065	0.025	0.064	0.047	0.018	0.038	0.026	0.008
	1	0.162	0.145	0.101	0.102	0.088	0.055	0.075	0.064	0.039	0.044	0.037	0.021
	2	0.158	0.144	0.100	0.098	0.088	0.056	0.075	0.069	0.045	0.048	0.044	0.029
	3	0.197	0.201	0.170	0.119	0.123	0.102	0.085	0.090	0.076	0.049	0.055	0.047
	4	0.211	0.229	0.209	0.120	0.136	0.126	0.081	0.096	0.092	0.042	0.056	0.057
2.0	0	0.186	0.167	0.125	0.125	0.109	0.073	0.096	0.083	0.056	0.061	0.052	0.034
	1	0.204	0.196	0.159	0.136	0.130	0.101	0.103	0.099	0.078	0.066	0.064	0.051
	2	0.179	0.172	0.134	0.115	0.110	0.082	0.091	0.090	0.071	0.062	0.063	0.051
	3	0.233	0.247	0.224	0.148	0.161	0.147	0.110	0.124	0.115	0.069	0.081	0.078
	4	0.248	0.278	0.269	0.149	0.177	0.175	0.106	0.132	0.135	0.060	0.083	0.090

* Values in bold indicate that RRMSE is smaller than TL-moments

4.2 Unknown Parent Distribution

The unknown parent distribution is more useful to see the effect on the estimation when the assumed distribution is totally different from the parent distribution function. In this study, the LN3 distribution function fitted to generate Generalized Pareto (GPA), Generalized Extreme Value (GEV) and Generalized Logistic (GLO) samples.

The quantile function $x(F)$ of the GPA, GEV, and GLO distributions are showed in Eqs. (28)-(30)

$$x(F) = \mu + \frac{\alpha}{\sigma} \left[1 - (1 - F)^\sigma \right] \tag{28}$$

$$x(F) = \mu + \frac{\alpha}{\sigma} \left[1 - (-\ln(F))^\sigma \right] \tag{29}$$

$$x(F) = \mu + \frac{\alpha}{\sigma} \left[1 - \left(\frac{1 - F}{F} \right)^\sigma \right] \tag{30}$$

The same parent distribution was applied as a parent distribution by using Monte Carlo simulation to see the performance of the L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 . The RBIAS and RRMSE values were obtained for quantile function $x(F)$ for $F = 0.90, F = 0.95, F = 0.98, F = 0.99,$ and $F = 0.995$, estimated by L-moments and TL-moments at samples $n = 15, 30, 50,$ and 100 . Box plot is used as a tool for grouping of the results based on statistical properties.

Box plot is a widely used graphical tool introduced by Tukey [30]. It is a simple plot of five quantities, namely, the minimum value, the lower quantile ($q_{0.25}$), the median ($q_{0.5}$), the upper quantile ($q_{0.75}$) and maximum value. The criteria for selecting a suitable TL-moments level are based on the minimum achieved median of RRMSE or RBIAS values, as well as the minimum dispersion in the median RRMSE or RBIAS values indicated by both ends of the box plot. It is noted that a smaller median dispersion in RRMSE or RBIAS values would indicate better integration of the TL-moments levels, so it should also be used as selection criterion.

Figures 1(a) – 1(d) present the box plots of RRMSE values on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1,$ and 0.3 for $n = 15$. For this small sample size, it gives almost similar result.

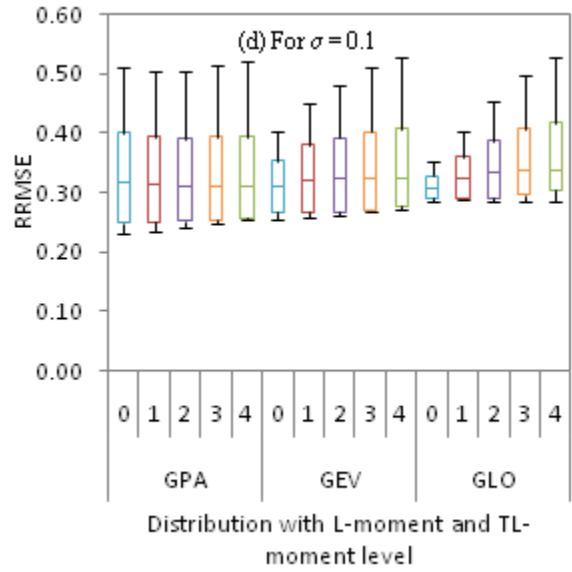
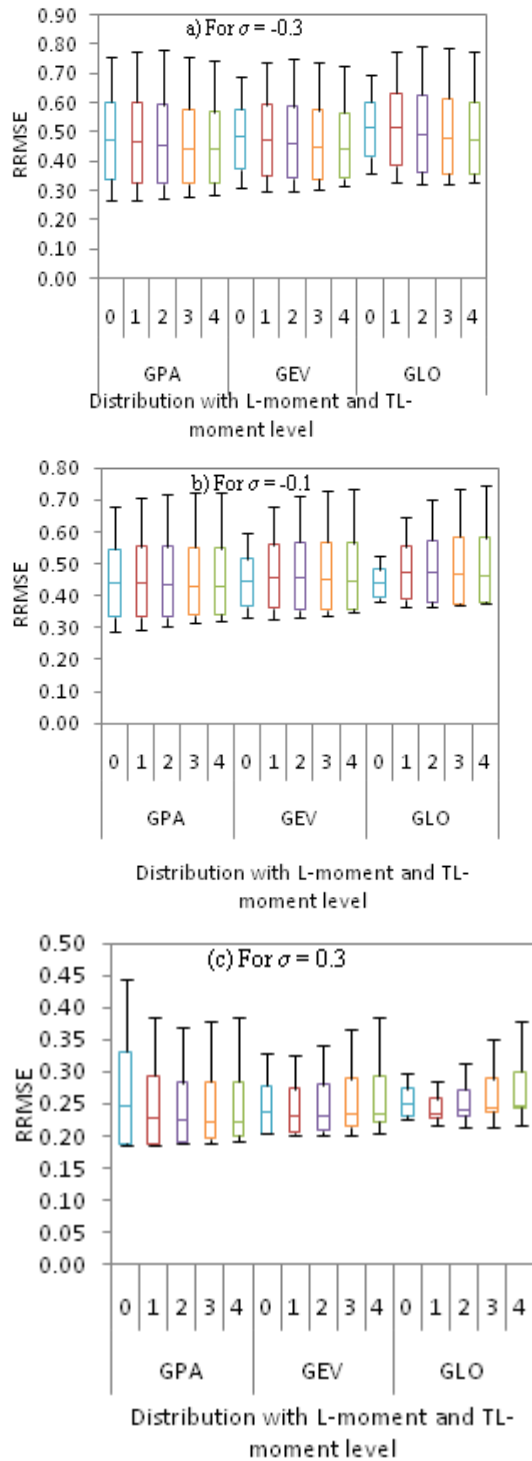
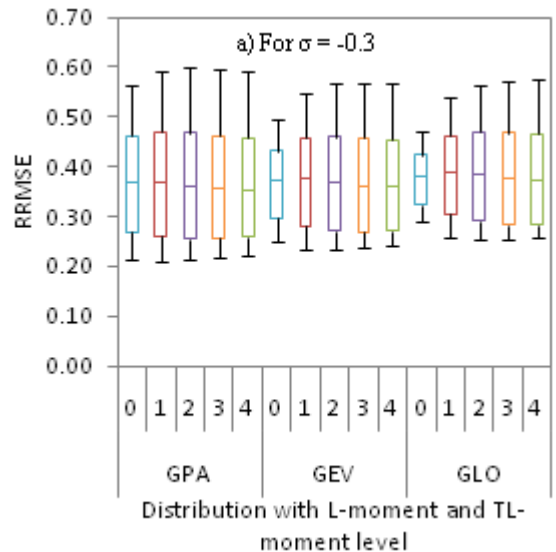
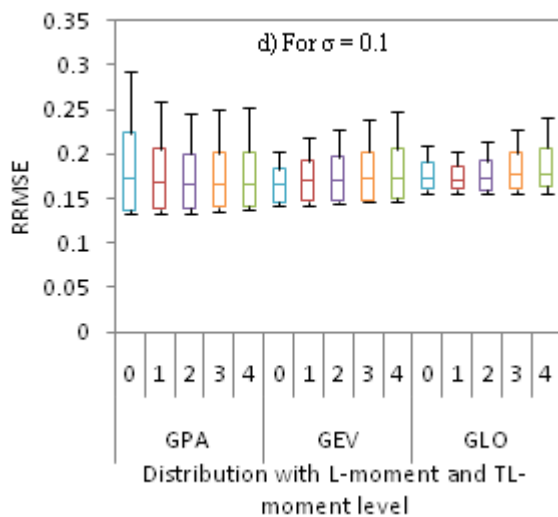
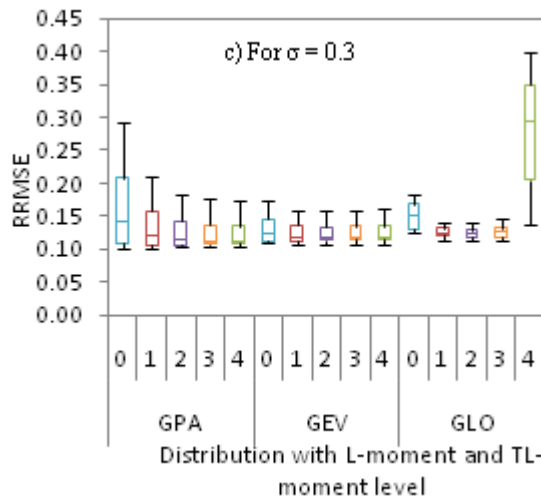
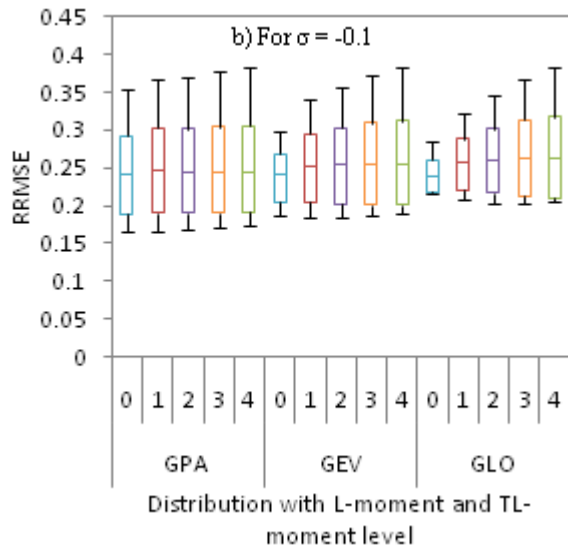


Figure 1 Box plots of RRMSE values on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1,$ and 0.3 for method L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 at $n = 15$

For sample size, $n = 50$ are shown in Figures 2(a) – 2(d). Almost all of the results give the same result for all three distributions (Figure 2(a)). The L-moment $\eta = 0$ level (Figure 2(b)) and TL-moment $\eta = 1$ level (Figure 2(d)) of the GLO distribution has minimum dispersion in RRMSE results. While, at sample size 50 and LN3 shape parameter, $\sigma = 0.3$, illustrate that TL-moment levels have minimum dispersion compare to L-moment $\eta = 0$ level for all three distributions.





Figures 3 and 4 represent the box plots of RBIAS values for sample size, $n = 15$ and 50 on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1,$ and 0.3 . TL-moment, $\eta = 1$ of the GLO distribution (Figures 3(a) – 3(d)) have minimum dispersion in RBIAS. However, L-moment $\eta = 0$ of the GEV distribution and TL-moment $\eta = 1$ of the GLO distribution give minimum median in RBIAS (Figure 3(a)). For sample size, $n = 50$, almost all TL-moment levels of the GLO distribution indicate the minimum dispersion in RBIAS (Figures 4(a) – 4(d)). TL-moment $\eta = 4$ of the GEV distribution (Figure 4(a)) provides minimum median RBIAS value of 0.0000 . The minimum dispersion RBIAS also can be seen at TL-moment levels of the GEV distribution in this figure. TL-moment $\eta = 2$ of the GEV distribution gives minimum median RBIAS value of -0.0003 (Figure 4(b)). While, TL-moment $\eta = 3$ and 4 level of the GLO distribution has minimum dispersion RBIAS (Figures 4(c) and 4(d)).

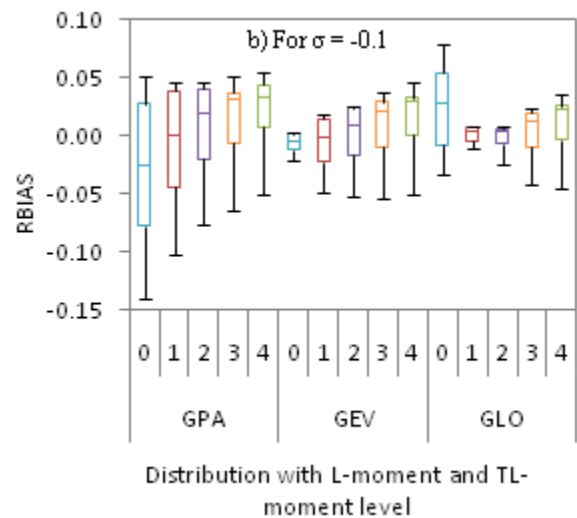
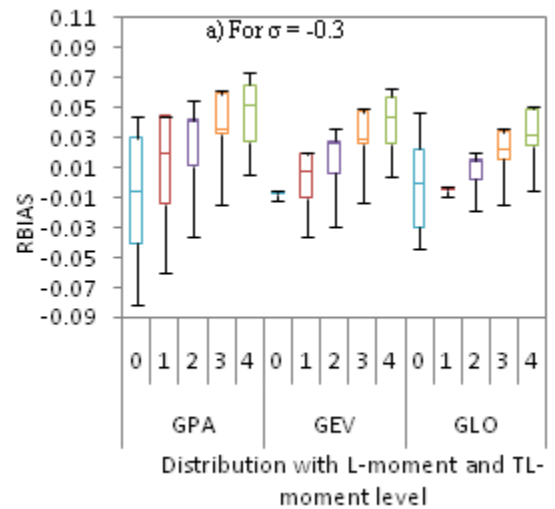


Figure 2 Box plots of RRMSE values on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1,$ and 0.3 for method L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 at $n = 50$

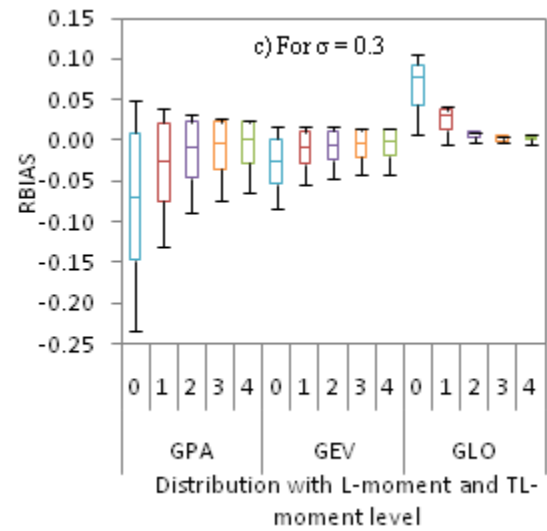
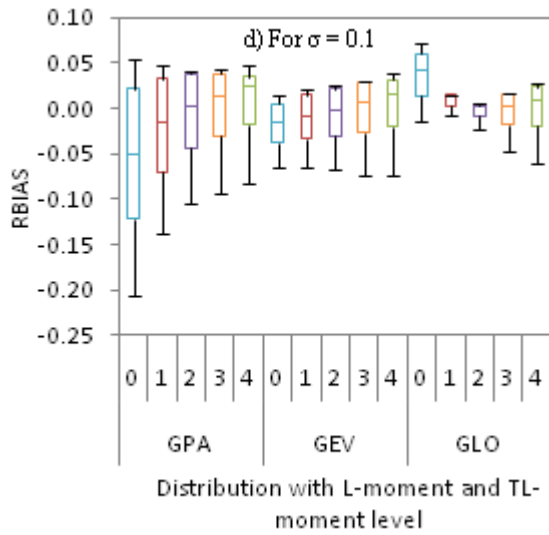
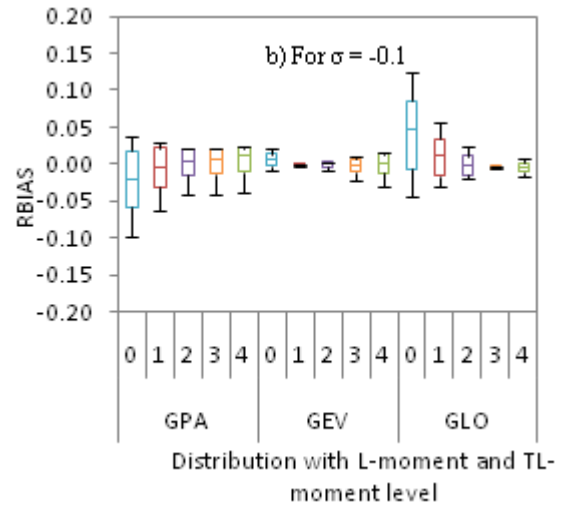
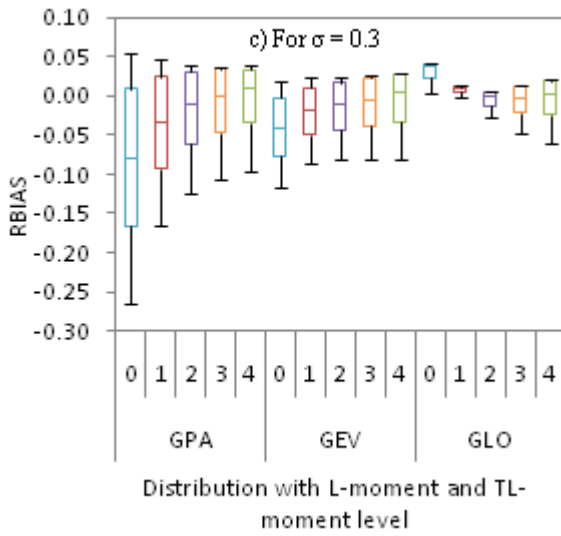


Figure 3 Box plots of RBIAS values on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1, \text{ and } 0.3$ for method L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3, \text{ and } 4$ at $n = 15$

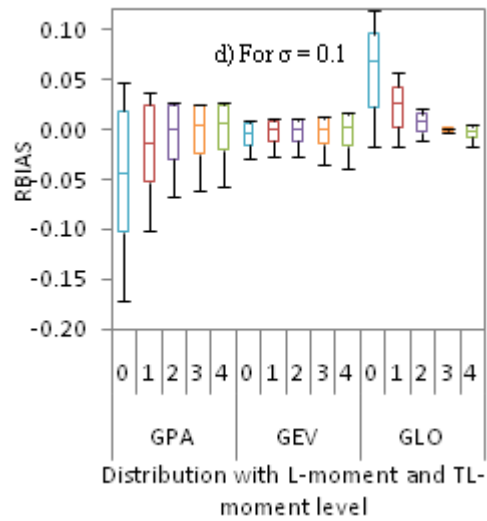
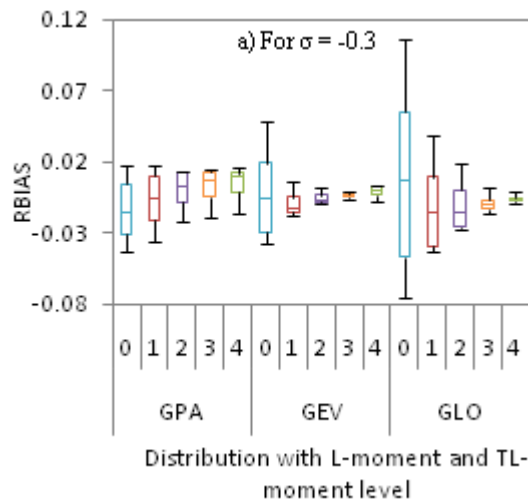


Figure 4 Box plots of RBIAS values on LN3 shape parameter, $\sigma = -0.3, -0.1, 0.1, \text{ and } 0.3$ for method L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3, \text{ and } 4$ at $n = 50$

4.3 Application To Hydrological Data

To show the different results between L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 for the LN3 distribution, stream flows data is needed. A set of annual maximum flow series for station 3813414 Sg. Trolak at Trolak and 3814416 Sg. Slim at Slim River located in Perak, Malaysia are used for this purpose. The data consists of 36 and 41 annual maximum stream flows data from 1960 until 2008 and 1966 until 2009 respectively. The data used in this study was provided by the Department of Irrigation and Drainage, Ministry of natural Resources and Environment, Malaysia.

The L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 are used to estimate the parameter of LN3 distribution. Table 5 presents the parameter estimates of the LN3 distribution. Figures 5 and 6 show five LN3 distribution curves fitted to the data series by using L-moment, $\eta = 0$ and the others by using TL-moments, $\eta = 1, 2, 3,$ and 4 .

Figure 5 represents the annual maximum stream flows data from Sg. Trolak at Trolak, Perak, Malaysia. For Sg. Trolak data, frequency curves obtained by the TL-moments, $\eta = 2, 3,$ and 4 lie much closer to the data than L-moment, $\eta = 0$ and TL-moments, $\eta = 1$ especially by the larger flows.

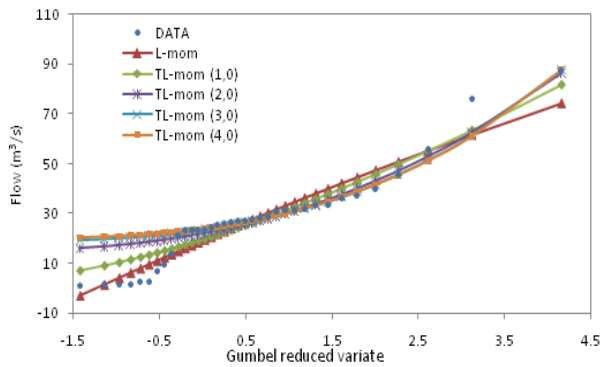


Figure 5 Fitting the LN3 distribution to annual maximum flows at Sg. Trolak at Trolak, Perak, Malaysia

For Sg. Slim data (Figure 6), all methods give almost fitted to the data. However, at the end of the flows, TL-moments are much closer to the data compared to L-moments. Hence, both data seem that TL-moments has good influenced by larger annual maximum flows. In contrast, L-moments ($\eta = 0$) shows the better trend by the small annual maximum flows.

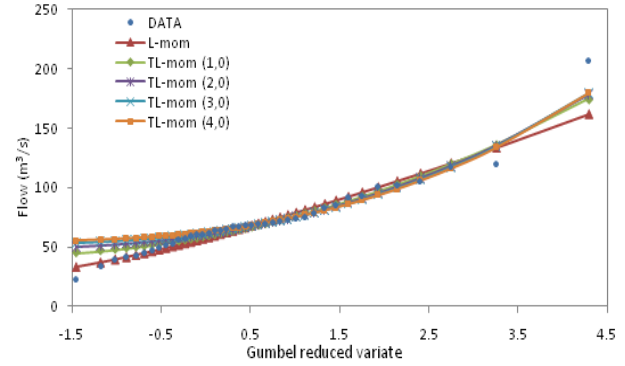


Figure 6 Fitting the LN3 distribution to annual maximum flows at Sg. Slim at Slim River, Perak, Malaysia

Table 5 Value of parameter estimates of the LN3 distribution

Station	η	σ	μ	α
Sg. Trolak	0	0.272	4.135	-38.038
	1	0.575	3.159	0.125
	2	0.916	2.302	14.525
	3	1.083	1.897	18.754
	4	1.12	1.805	19.636
Sg. Slim	0	0.502	3.86	17.745
	1	0.784	3.165	40.791
	2	0.914	2.851	47.811
	3	0.999	2.645	51.826
	4	1.049	2.518	54.233

5.0 CONCLUSION

A robust generalization of population and sample L-moments was defined by Elamir and Seheult [12] as a definition of population trimmed L-moments (TL-moments) and corresponding sample TL-moments. The L-moments, $\eta = 0$ and TL-moments $\eta = 1$ to $\eta = 4$ are used to estimate the regional parameters of the LN3 distribution. We used the combination of PWM and L-moment method to formulate the estimation of the LN3 distribution.

Monte Carlo simulation study is conducted to compare the performance of L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3$ and 4 for the LN3 distribution. For known parent distribution, the RRMSE result of TL-moment method is leading when LN3 shape parameter, σ becomes larger either in small sample size or big sample size. The quantile estimates for L-moment and TL-moments are becoming more unbiased for most quantiles when σ becomes smaller. However, when σ becomes larger, the quantile estimator becomes more positively biased almost for all sample sizes. For unknown parent distribution, LN3 distribution function is fitted to generate GPA, GEV, GLO samples. Box plot is used as a tool for grouping the results based on statistical properties. The criteria for selecting a suitable TL-

moments level are based on the minimum achieved median of RRMSE or RBIAS values, as well as the minimum dispersion in the median of RRMSE or RBIAS values indicated by both ends of the box plot. From the results, most box plots produced similar results in RRMSE with small sample size. With a large sample size, positive shape parameter of LN3 gives the minimum dispersion in RRMSE for almost all level of L-moment and TL-moment. For RBIAS box plots, TL-moment level give the better result by showing the minimum median and minimum dispersion in RBIAS mostly for GLO distribution.

We applied the method of L-moment, $\eta = 0$ and TL-moment, $\eta = 1, 2, 3,$ and 4 for the LN3 distribution in annual maximum flow series Sg. Trolak and Sg. Slim where are located in Perak, Malaysia. The TL-moments method give the good fit to the data by larger annual maximum flows.

Acknowledgement

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