MATHEMATICAL MODELS FOR FREE AND MIXED CONVECTION BOUNDARY LAYER FLOWS OF MICROPOLAR FLUIDS

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To my husband, parents and family –
thank you for everything

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ABSTRACT

The phenomena of free and mixed convection are encountered in many industrial and engineering applications, for example, in the cooling of electronic equipment, materials processing and drilling operations. Free convection has also been used to explain the connection between skin disease and respiratory disease such as eczema and asthma. In this study, the mathematical models for steady laminar free and mixed convection boundary layer flows over a horizontal circular cylinder and a sphere immersed in an incompressible micropolar fluid are developed. The theory of micropolar fluid was proposed, as the classical Navier-Stokes theory is inadequate to describe most industrial fluids. Examples of micropolar fluids include polymeric fluids and colloidal suspensions that take into account the microscopic effects arising from the local structures and micromotions of the fluid elements. Both isothermal and non-isothermal boundary conditions are considered. The governing nonlinear partial differential equations are first transformed using an appropriate nonsimilar transformation before they are solved numerically using the Keller-box method, an unconditionally stable implicit finite-difference scheme. Numerical results presented include the velocity, temperature and angular velocity profiles as well as the fluid flow and heat transfer characteristics, for a range of the material parameter or the vortex viscosity K, the Prandtl number Pr, and the mixed convection parameter λ. The numerical codes in the form of software packages have been developed using Matlab®. The packages and numerical results presented constitute an invaluable reference against which other exact or approximate solutions can be compared in the future.

ABSTRAK

Fenomena olakan bebas dan olakan campuran sering ditemui dalam kebanyakan penggunaan industri dan kejuruteraan, misalnya dalam penyejukan alat elektronik, pemprosesan bahan dan operasi penggerudian. Di samping itu, olakan bebas juga telah digunakan bagi menerangkan hubungan antara penyakit kulit seperti ekzema dengan penyakit sistem pernafasan seperti lelah. Dalam kajian ini, model-model matematik bagi olakan bebas dan olakan campuran yang mantap dan berlamina terhadap silinder bulat melintang dan sfera dalam bendalir mikropolar tak mampat dipertimbangkan. Teori bendalir mikropolar dicadangkan kerana teori klasik Navier-Stokes tidak sesuai untuk menerangkan kebanyakan bendalir industri, contohnya bendalir polimer dan ampaian koloid. Teori ini mempertimbangkan kesan mikroskopik yang terhasil daripada struktur-struktur setempat dan pergerakan mikro unsur-unsur bendalir tersebut. Syarat-syarat sempadan isoterma dan bukan isoterma dipertimbangkan. Persamaan pembezaan separa tak linear yang tertakluk dijelmakan menggunakan jelmaan tak serupa yang sesuai. Persamaan-persamaan ini kemudiannya diselesaikan secara berangka menggunakan kaedah kotak-Keller. Kaedah kotak-Keller merupakan suatu skim beza terhingga tersirat yang stabil tanpa syarat. Penyelesaian berangka yang diperoleh mengandungi profil-profil halaju, suhu dan halaju sudut, di samping ciri-ciri aliran bendalir dan pemindahan haba bagi julat parameter bahan atau kelikatan vorteks K, nombor Prandtl, Pr dan parameter olakan campuran λ. Kod berangka dalam bentuk pakej perisian telah dibina menggunakan Matlab®. Pakej ini beserta penyelesaian berangka yang dihasilkan boleh dijadikan sumber rujukan berharga untuk tujuan penyemakan keputusan penyelesaian tepat atau penyelesaian hampir pada masa hadapan.

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LIST OF SYMBOLS

a	-	Radius of the cylinder (or sphere)
a, b	-	Lower and upper bounds of the domain of mixed convection regime, respectively
C_f	-	Local skin friction coefficient
f / F	-	Reduced stream function
g	-	Acceleration due to gravity
Gr	.	Grashof number
h / G	-	Reduced microrotation function
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	-	Unit vectors
J	-	Microinertia density
k	-	Thermal conductivity
K	-	Vortex viscosity or material parameter of micropolar fluid
m	-	Power index
n	-	Constant
N	-	Non-dimensional microrotation component normal to the (x,y)-plane / non-dimensional angular velocity of micropolar fluid
Nu	-	Nusselt number
p	-	Pressure
Pr	-	Prandtl number
q_w	.	Surface heat flux
Q_w	-	Local heat transfer coefficient
$r(x)$	-	Radial distance from symmetrical axis to surface of the sphere
Re	-	Reynolds number
T	-	Fluid temperature

T_0	-	Temperature of outside surface of cylinder
u, v	-	Non-dimensional velocity components along x and y directions, respectively
$u_e(x)$	-	Non-dimensional velocity outside boundary layer
U_∞	-	Free stream velocity
$\mathbf{V}$	-	Velocity vector
x, y	-	Non-dimensional Cartesian coordinates along the surface of the cylinder (or sphere) and normal to it, respectively

Greek symbols

α	-	Thermal diffusivity
β	-	Thermal expansion coefficient
γ	-	Spin gradient viscosity
κ	-	Vortex viscosity
λ	-	Mixed convection parameter
μ	-	Dynamic viscosity
ν	-	Kinematic viscosity
θ	-	Non-dimensional temperature
ρ	-	Fluid density
ρ_∞	-	Constant local density
τ_w	-	Reduced skin friction parameter
ψ	-	Non-dimensional stream function
$\overline{\nabla}^2$	-	Laplacian operator

Superscripts

$'$	-	Differentiation with respect to y or η
$-$	-	Dimensional variables
$\wedge$	-	Non-dimensional variables

Subscripts

w	-	Condition at the wall
∞	-	Ambient / free stream condition

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

The convective mode of heat transfer is generally divided into two basic processes, natural or free convection, and forced convection. In several practical applications, temperature differences exist in the boundary region near a heated or cooled surface. The temperature differences cause density gradients in the fluid medium, and in the presence of a body force such as gravity, free convection effects arise (Kakac, 1980). The density difference gives rise to buoyancy forces. If the motion of the fluid arises from an external agent, then the process is termed forced convection, and the externally imposed flow is generally known. Thus, in any forced convection situation, free convection effects are also present under the presence of gravitational body forces. In addition, when the effect of the buoyancy force in forced convection, or the effect of forced flow in free convection becomes significant, then the process is called mixed convection flows, or combined forced and free convection flows. The effect is especially pronounced in situations where the forced fluid flow velocity is low and/or the temperature difference is large. In mixed convection flows, the free convection effects and the forced convection effects are of comparable magnitude (Pop and Ingham, 2001).

The domain of mixed convection regime is generally defined as the region $a \le \mathrm{Gr}/\mathrm{Re}^m \le b$, where Gr and Re are the Grashof and Reynolds numbers, respectively, which will be described further in the chapter, while a and b are the lower and upper bounds of the domain, respectively. The buoyancy parameter $\mathrm{Gr}/\mathrm{Re}^m$, provides a measure of the influence of free convection in comparison with that of forced convection on the flow. The power m depends on the flow configuration and the surface heating condition. Outside the mixed convection region ($a \le \mathrm{Gr}/\mathrm{Re}^m \le b$), either the pure forced convection or the pure free convection analysis can be used to describe accurately the flow or the temperature field. Forced convection is the dominant mode of transport when $\mathrm{Gr}/\mathrm{Re}^m \to 0$, whereas free convection is the dominant mode when $\mathrm{Gr}/\mathrm{Re}^m \to \infty$ (see Chen and Armaly (1987)).

Convective heat transfer can also be classified as either bounded or unbounded, which is more commonly known as internal or external flow, respectively. Both the free and mixed convection processes may be divided into external flow over immersed body (such as flat plates, cylinders and spheres), and internal flow in ducts (such as pipes, channels and enclosures). The resultant flow can further be classified as either laminar (stable) or turbulent (unstable) flow. The laminar flow is smooth, with a particle of fluid moving steadily in a smooth line parallel to the surfaces, and a thin layer of fluid then moves as a lamination. On the other hand, the turbulent flow is described as an erratic and chaotic flow with a particle of fluid moving unsteadily in an unpredictable zigzag path. Turbulent flow is generally expected to occur when Reynolds number is high while laminar flow is when Reynolds number is low (Burmeister, 1993).

Although extensive research work has been devoted to heat transfer in viscous (Newtonian) fluids, more recently, research in non-Newtonian fluids has gained momentum as well. This is due to the increasing importance in the processing industries and elsewhere of materials whose flow behaviour in shear cannot be characterized by Newtonian relationships. Free convection in non-Newtonian fluids has become vital for the design of equipment and for tackling problems related to the handling and processing of such fluids which are commonly handled by the food, paint, polymer and pharmaceutical industries. One particular non-Newtonian fluid that shall be considered in this study is the micropolar fluid.

The present study considers the problems of free and mixed convection boundary layer flows over circular cylinders and spheres in micropolar fluids. In this study, the flow is assumed to be laminar, the external flow is considered and steady state prevails. Two cases of boundary condition are discussed: (i) constant surface temperature and (ii) constant surface heat flux. The analysis include: (1) formulation of the mathematical models to obtain the governing boundary layer flow and heat transfer equations for the new models; (2) nonsimilar boundary layer transformation; and (3) numerical computation using a finite-difference scheme. The scheme employed is the Box method developed by Keller (1970, 1971), and throughout the whole course of this study, the main references for the Keller-box method are the books by Cebeci and Bradshaw (1977, 1988) and Na (1979). It is worth mentioning that, besides the existing work using Keller-box method in the literature, this method has also been proven to be good and efficient as the present author has used it successfully in solving various convective boundary layer problems (Roslinda Nazar and Norsarahaida Amin, 2001, 2002, 2003a, 2003b; Nazar *et al.*, 2002a – 2002e, 2003a – 2003g).

In the next section, we present the objectives and scope of this present study, followed by the thesis outline, which describes the content of the entire thesis. Further, some discussions on the significance of free and mixed convection, boundary layer theory and micropolar fluid are presented in Sections 1.4, 1.5 and 1.6, respectively. Next, there will be a literature review section, and finally, the basic governing equations and derivations are presented in the last section.

1.2 Objectives and Scope

The objectives of the present study are to construct mathematical models, to carry out mathematical formulations and analyses, and to develop numerical algorithms for the computations of the following four main problems:

Problem 1: Free convection boundary layer over a horizontal circular cylinder in a micropolar fluid

Problem 2: Free convection boundary layer about a sphere in a micropolar fluid

Problem 3: Mixed convection boundary layer over a horizontal circular cylinder in a micropolar fluid

Problem 4: Mixed convection boundary layer about a sphere in a micropolar fluid

The scope of study is limited to problems involving steady, two-dimensional laminar free and mixed convection boundary layer flows over horizontal circular cylinders and spheres, immersed in viscous and incompressible micropolar fluids, with two types of boundary conditions, namely the constant surface temperature and constant surface heat flux. These problems are formulated using the nonsimilar transformation and solved numerically using the Keller-box method.

1.3 Thesis Outline

This thesis is divided into seven chapters including this introductory chapter. Chapters 1 and 2 should be regarded as preliminaries with general introduction and formulation. In this study, the problems of steady laminar free and mixed convection boundary layer flows over horizontal circular cylinders and spheres with constant surface temperature and constant surface heat flux are considered in micropolar fluids. These problems are generally divided into four main chapters, namely Chapters 3 through 6, which discuss the four main problems mentioned in the previous section. Basically, each chapter discusses two cases of boundary conditions as mentioned above.

All of these problems are solved numerically using the Keller-box method. Full discussions and details of the numerical method, the Keller-box method are given in Chapter 2, which are presented and described specifically for the first main problem in micropolar fluid, the free convection boundary layer flow over an isothermal horizontal circular cylinder in a micropolar fluid. Stepwise development of the method is presented. This method has been found to be suitable and flexible to deal with the problems of free and mixed convection. The Keller-box method used in this study is programmed in Matlab® 5.3.1. The complete program of the specific problem discusses in Chapter 2 is given in Appendix B. The next following chapters of this thesis are Chapters 3 to 6, which discuss the four main problems as outlined below.

In Chapter 3, we discuss the first problem in micropolar fluid. The problem is free convection boundary layer over a horizontal circular cylinder in a micropolar fluid. This chapter will be divided into four main sections where the first section is the introduction of the problem, second and third section each will describe in details the problems engaging in two different boundary conditions, namely the constant surface temperature and constant surface heat flux, respectively. For each of these two sections, there are two subsections, which are the Basic Equations and the Results and Discussions. The final section contains the Conclusions of this main problem.

Discussions on relevant physical quantities such as the local skin friction coefficient, local heat transfer coefficient and local wall temperature are presented in the Results and Discussions subsections. Some discussions on temperature, velocity and angular velocity profiles are also included. Our present results when the material parameter $K = 0$ (Newtonian fluid) are compared with existing results from the literature for similar problem in viscous fluid and the agreement are excellent, and thus, we proceed to get the new results for other values of K ($K \neq 0$) in micropolar fluid. All related figures are presented, and some results are also given in the form of tables in all the chapters. Such tables are very important and they can serve as a reference against which other exact or approximate solutions can be compared in the future.

On the other hand, in Chapter 4, we discuss free convection boundary layer problems in a micropolar fluid as well, but the geometry of the problem is about a sphere. The division of sections and subsections are similar to those in Chapter 3.

Similar to Chapter 3, we also begin the discussion by comparing our present results for the material parameter K = 0 (Newtonian fluid) with existing results from the literature for similar problem in a viscous fluid and the agreement are found to be excellent. Other numerical results as in Chapter 3 are also presented.

In Chapters 3 and 4 we discuss free convection problems, while in the next two chapters we discuss mixed convection problems. In Chapter 5, the problem of steady laminar mixed convection boundary layer flow over a horizontal circular cylinder in a micropolar fluid is studied. All the sections and subsections are divided similar to previous chapters. In addition, for the mixed convection problems, we also have the results for the variations of separation points given in the form of tables and figures, where the boundary layer separations are indicated.

Finally, in Chapter 6, we discuss problems of mixed convection boundary layer in micropolar fluid as well, but the geometry of the problem is about a sphere. This chapter discusses problem of laminar mixed convection boundary layer about a sphere in a micropolar fluid with constant surface temperature and heat flux. The division of sections and subsections are also similar to previous chapters and likewise, numerical results for the flow and heat transfer characteristics are presented for each case. The results for variations of the boundary layer separations are also presented.

The final and concluding chapter, Chapter 7, contains some concluding remarks, summary of research as well as some suggestions for future research. All the references in this thesis are listed in the reference section at the end of Chapter 7. There are two appendices in this thesis, Appendices A and B. These appendices represent the notation of the symbols and variables used in the Matlab® program and the Matlab® program for the specific problem discusses in Chapter 2, respectively.

1.4 Significance of Free and Mixed Convection

The range of free convection flows that can occur in nature and engineering, has been extensively reviewed by Gebhart *et al.* (1988), and Pop and Ingham (2001). Free convection flow is a significant factor in, for example, aeronautics, cooling of electronic components, designing of chemical processing equipment, safety reactor, combustion flame and solar collectors. In particular, it has been ascertained that free convection can induce the thermal stresses, which lead to critical structural damage in the piping systems of nuclear reactors. The buoyant flow arising from heat rejection to the atmosphere, heating of rooms, fires, and many other such heat transfer processes, both natural and artificial, are other examples of free convection flows. In addition, the central receiver of solar power plants is a classic example of combined free and forced convection or mixed convection. The receiver is heated by solar radiation and its surface temperature is much higher than the ambient temperature. Significant free convection is induced by the density stratification of air in the thermal boundary layer. Simultaneously, a breeze may pass by the receiver and disturb the free convection around the receiver.

Further, it is noted that a considerable amount of work has been devoted to the study of free or mixed convection over flat plates. Recently, convective heat transfer about heated bodies with different geometry has begun to attract a great deal of attention, owing to the fact that cylindrical and spherical shapes of canisters have been proposed for nuclear waste disposal in sub-seabeds (Minkowycz, 1985). The study of convective flow over fixed two-dimensional and axisymmetric bodies of arbitrary shape in an infinite fluid medium constitutes an important heat transfer problem from the standpoint of theoretical and engineering applications. Free convection over a circular cylinder or a sphere represents also an important problem, which is related to numerous engineering applications. An exact analytical solution is still out of reach due to the nonlinearities in the Navier-Stokes and energy equations. The earliest attempts to compute this problem involved solving the simplified boundary layer equations.

Mixed convection flow past horizontal cylinders has received much attention and is important in situations encountered in the areas of geothermal power generation and drilling operation, when the free stream velocity and the induced buoyancy velocity are of comparable order. It continues to be one of the most important problems, also due to its fundamental nature as well as many engineering applications. In spite of the fact that a good number of theoretical and experimental studies were carried out in the past on mixed convection flow, it seems that most of these studies are limited to cases in which the forced flow is directed upward (assisting flow). An immense body of literature exists for the case of Newtonian fluids (Gebhart *et al.*, 1988; Pop and Ingham, 2001).

1.5 Boundary Layer Theory

The boundary layer theory was proposed by Ludwig Prandtl in 1904. The main idea is to divide the flow into two major parts. The larger part concerns a free stream of fluid, far from any solid surface, which is considered to be inviscid. The smaller part is a thin layer adjacent to the solid surface in which the effects of viscosity are felt (Acheson, 1990). This thin layer where friction effects cannot be ignored is called the boundary layer. Through experimental observations, Prandtl found that large velocity gradients normal to the streamlines occur only in regions close to the surface.

Prandtl concluded that it might be sufficient in an analysis of a flow field to consider action of viscosity within these boundary layers, whereas the flow outside the boundary layers may be considered inviscid. He then proceeded to simplify the conservation equation by estimating the order of magnitude of the various terms in the conservation equations, and thus he derived the so-called boundary layer equations (Eckert and Drake, 1972). Analytical treatment of the Navier-Stokes equations present great difficulties even in the case of steady two-dimensional incompressible flows. Only a limited number of exact solutions are known to exist for some special cases of these equations. An important contribution by Prandtl was to show that the Navier-Stokes equations can be simplified to yield an approximate set of boundary layer equations (Bejan, 1984).

There are many reasons why the boundary layer theory is used very frequently in solving fluid flow and heat transfer problems (see Bejan (1984) and Cebeci and Bradshaw (1988). The most important reason, at least for this present study, is because the boundary layer equations are parabolic while the full Navier-Stokes equations are elliptic or sometimes, even hyperbolic, which are of considerable complexity. Thus, parabolic partial differential equations can be solved much easier. Further, the boundary layer theory also gives more information about the flow separation from the surface of a body than full Navier-Stokes equations. However, the boundary layer equations are available only up to the separation point and beyond this point, the full Navier-Stokes equations have to be solved with much complexity.

1.6 Micropolar Fluid

The essence of the theory of micropolar fluid flow lies in the extension of the constitutive equations for Newtonian fluids, so that more complex fluids such as particle suspensions, liquid crystals, animal blood, lubrication and turbulent shear flows can be described by this theory. The theory of micropolar fluid, first proposed by Eringen (1966), is capable of describing such fluids. This theory has generated much interest, and many classical flows are being re-examined to determine the effects of the fluid microstructure. This theory is a special class of the theory of microfluids, in which the elements are allowed to undergo only rigid rotations without stretch. Practically, the theory of micropolar fluid requires that one must add a transport equation representing the principle of conservation of local angular momentum to the usual transport equations for the conservation of mass and momentum, and also additional local constitutive parameters are introduced.

Thermal convection flows of these fluids along a solid surface are also important due to their applications in a number of processes that occur in the modern industry. Such applications include the extrusion of polymer liquids, solidification of liquid crystals, animal blood, etc., for which the classical Navier-Stokes theory is inadequate. The key points to note in the development of Eringen's microcontinuum mechanics are the introduction of new kinematic variables, e.g. the gyration tensor and microinertia moment tensor, and the addition of the concept of body moments, stress moments, and microstress averages to classical continuum mechanics. These special features of micropolar fluids were discussed in a comprehensive review paper of the subject and application of micropolar fluid mechanics by Ariman *et al.* (1973). The recent books by Łukaszewicz (1999) and Eringen (2001) provide a useful account of the theory and extensive surveys of the literature of the micropolar fluid theory.

1.7 Literature Review

The discussion on the literature review will be presented in the next four sub-sections with regards to the four problems in micropolar fluids as mentioned in Section 1.2. In addition, the literature review related to the numerical method used in this study, the Keller-box scheme, will be presented in the last subsection.

1.7.1 Free Convection over Horizontal Circular Cylinders

The theory of micropolar fluid in which the local structure and micromotions of the fluid elements give rise to the microscopic effects, has been formulated by Eringen (1966). Later, Eringen (1972) has generalized the theory of micropolar fluid to include thermal effects, as well. Willson (1969) has introduced the problem of boundary layers in micropolar liquids. The concept of boundary layer theory in micropolar fluid was first introduced by Willson (1970). In the same year, Peddieson and McNitt (1970) applied the micropolar boundary layer theory to the problems of steady stagnation point flow, steady flow over a semi-infinite flat plate, and impulsive flow past an infinite flat plate.

On the other hand, Nath (1975, 1976) developed similar and non-similar solutions, respectively, for the incompressible laminar boundary layer in micropolar fluids and Ahmadi (1976) also considered the self-similar solution of incompressible micropolar boundary layer flow over a semi-infinite flat plate. Gorla (1983) investigated the steady boundary layer flow of a micropolar fluid at a two-dimensional stagnation point on a moving wall. Kumari and Nath (1984) studied the unsteady flow near a plane stagnation point when the free stream velocities vary arbitrary with time. Pop *et al.* (1998a) studied the convective wall plume in micropolar fluids, while Mohammadein and Gorla (2001) described the problem of heat transfer in a micropolar fluid over a stretching sheet.

Research on free convection boundary layer flows over flat plates in micropolar fluids has been conducted earlier by Jena and Mathur (1981, 1982), Gorla *et al.* (1983, 1995) as well as Chiu and Chou (1993, 1994), and recently by Rees and Pop (1998) and Gorla *et al.* (1998a, 1998b) for the cases of vertical flat plates. The Blasius boundary layer flow of a micropolar fluid over a flat plate is considered by Rees and Bassom (1996). Walicki and Walicka (1979), and Walicka (2001) considered the flow of micropolar fluid in a slot between surfaces of revolution.

Literature on free convection boundary layer flow over circular cylinders and spheres is not studied as extensive as the boundary layer flow over flat plates. Bhattacharyya and Pop (1996) studied free convection boundary layer over a horizontal cylinder of elliptic cross-section in micropolar fluid, which is an extension of the problem studied by Merkin (1977b) in viscous fluid. The boundary layer flow over a heated horizontal plane has been analyzed by Amin and Riley (1990), while Merkin

(1976) considered the case of free convection boundary layer on an isothermal horizontal circular cylinder in a viscous fluid. It appears that Merkin (1976) was the first to present a complete solution of this problem for a classical (Newtonian) fluid using Blasius and Gortler series expansion methods along with an integral method and a finite difference scheme. Merkin and Pop (1988) studied the free convection boundary layer on a horizontal circular cylinder with constant heat flux in a viscous fluid, while Ingham and Pop (1987) studied free convection about a heated horizontal cylinder in a porous medium.

Motivating by the studies conducted by Merkin (1976), and Merkin and Pop (1988) in viscous fluids, the present study considers laminar free convection boundary layer flow over a horizontal circular cylinder in a micropolar fluid, with constant surface temperature and constant surface heat flux.

1.7.2 Free Convection about Spheres

Free convection flow from general bodies in the existence of a boundary layer in a micropolar fluid has been considered by Gorla and Takhar (1987) and Takhar *et al.* (1988) for cases past slender bodies, Chiu and Chou (1993), and Pop *et al.* (1998b) for cases along wavy surfaces. On the other hand, Char and Chang (1995) studied the laminar free convection flow of a micropolar fluid from a curved surface.

Free convection from a sphere represents also an important problem, which is related to numerous engineering applications. An exact analytical solution is still out of reach due to the nonlinearities in the Navier-Stokes and energy equations. Therefore, the problem of free convection from a sphere has received relatively little attention. In the present study, a numerical study is considered for the problem of steady boundary layer free convection from a solid sphere immersed in a micropolar fluid with uniform surface temperature and uniform surface heat flux. The obtained results are compared with those reported by Huang and Chen (1987) for the corresponding problem in a classical (Newtonian) fluid and the agreement is excellent.

1.7.3 Mixed Convection over Horizontal Circular Cylinders

The problem of mixed convection boundary layers on vertical flat plates in viscous fluids for both isothermal and uniform heat flux cases has received much attention in the past. Gryzagoridis (1975) presented a good description of the previous work on the subject. It appears that the first theoretical study on mixed convection flow from general bodies in the existence of a boundary layer has been carried out by Acrivos (1966). Later, Joshi and Sukhatme (1971) used series solution method to study the boundary layer flow of this problem for both cases of assisting and opposing flow. On the other hand, Nakai and Okazaki (1975) studied the problem of mixed convection from a circular cylinder for the cases when both the Reynolds number Re and Grashof number Gr, are very small and also when either forced convection or free convection is dominant. Next, Sparrow and Lee (1976) considered the assisting flow regime of this problem.

Merkin (1977a) has studied the same problem by obtaining a numerical solution to the boundary layer equations based on the assumption that $Re \gg 1$ and $Gr \gg 1$ with $Pr = 1$. The solution was again restricted to the region preceding the point of boundary layer separation since the boundary layer equations are not valid beyond that point. Further, Badr (1983, 1984) studied the mixed convection heat transfer from an isothermal horizontal circular cylinder based on the solution of the full Navier-Stokes and energy equations. Recently, Aldoss *et al.* (1996) and, Aldoss and Ali (1997) investigated the effect of a radial magnetic field on the flow and heat transfer characteristics of mixed convection boundary layer flow from a horizontal circular cylinder with variable surface temperature. The results were obtained numerically using the local nonsimilarity method and the coordinate perturbation method. On the other hand, the unsteady boundary layer flow over a horizontal circular cylinder has been studied by Katagiri and Pop (1979) as well as Ingham and Merkin (1981).

The theory of micropolar fluid has generated a lot of interests and many flow problems have been studied. However, studies of the external convective flows of micropolar fluid have focused mainly on either pure forced or pure free convection problems (see Ahmadi (1976), Mathur *et al.* (1978), Gorla *et al.* (1998a, 1998b), Pop *et al.* (1998b), Rees and Pop (1998), Hossain *et al.* (1999, 2000), Cheng and Wang (2000), Nazar *et al.* (2002a – 2002d)). Relatively few studies have considered problems of combined forced and free convection (mixed convection) in micropolar fluids (see Gorla (1988, 1992), Kumari and Nath (1989), Arafa and Gorla (1992), Wang (1993), Hossain and Chowdhury (1998), Hossain *et al.* (1995)). Mixed or combined convection is a situation where both forced and free convection effects are of comparable order. Recent examples of application can be found in areas of combustion flames as well as building energy conservation (Hassanien and Gorla, 1990).

Mixed convection flow from horizontal cylinders constitutes an important heat transfer problem from the standpoint of engineering applications and numerical analyses. An immense body of literature exists for the case of Newtonian fluids (Gebhart *et al.*, 1988; Pop and Ingham, 2001). However, relatively fewer investigations of mixed convection have been conducted, especially for the case of micropolar fluids. Recently, Mansour and Gorla (1999) have considered the problem of MHD mixed convection flow near the lower stagnation point of a horizontal circular cylinder immersed in a micropolar fluid using the similarity equations.

Study on mixed convection about a nonisothermal horizontal cylinder and a sphere in a porous medium has been conducted by Minkowycz (1985). Research on mixed convection boundary layer flows over flat plates in micropolar fluids have been conducted by Gorla (1988) on a vertical flat plate with isothermal wall and constant surface heat flux. Gorla (1992) studied mixed convection from a vertical surface with uniform heat flux, as well as on boundary layer flow over a horizontal plate (Gorla, 1995). Recently, Hsu *et al.* (2000) considered the problem of mixed convection of micropolar fluid along a vertical wavy surface, while more recently, Chu *et al.* (2002) also studied the problem of mixed convection along a vertical wavy surface in micropolar fluids, but with a discontinuous temperature profile.

Motivated by the work of Merkin (1977a) and Nazar *et al.* (2003c) for mixed convection boundary layer problems over a horizontal circular cylinder in viscous fluids, with constant temperature and constant heat flux, respectively, and also the work by Hassanien and Gorla (1990) in mixed convection boundary layer flow of a micropolar fluid near a stagnation point on a horizontal cylinder, and Hassanien (1994) on a vertical slender cylinder in a micropolar fluid, this present study is going to tackle the problem of mixed convection boundary layer flows over a horizontal circular cylinder in micropolar fluids with constant wall temperature and heat flux.

1.7.4 Mixed Convection about Spheres

The problem of mixed, (forced and free) convection about a solid sphere in a viscous and incompressible fluid has received relatively little attention. To our best knowledge, the only such studies which have been reported are the experimental work of Yuge (1960) and Klaychko (1963), and the analytical work of Hieber and Gebhart (1969). These studies, both experimental and analytical were conducted under the action of very small Reynolds and Grashof numbers. Chen and Mucoglu (1977) and Mucoglu and Chen (1978) have later studied the mixed convection over a sphere with uniform surface temperature and uniform surface heat flux for very large Reynolds and Grashof numbers using the boundary layer approximation. It is also worth mentioning the papers by Dennis and Walker (1971) who studied the steady forced convection flow past a sphere at low and moderate Reynolds numbers and Alassar and Badr (1997) who considered the oscillating viscous flow over a sphere. Other papers of interest for practical applications in engineering heat transfer on this topic are those by Le Palec and Daguenet (1987), El-Shaarawi *et al.* (1990), Jia and Gogos (1996) and most recently, Antar and El-Shaarawi (2002).

It appears that Lien and Chen (1987) were the first to study the steady mixed convection boundary layer flow problem from a sphere in a micropolar fluid, and Wang and Kleinstreuer (1988) generalized the paper by Lien and Chen (1987) to two-dimensional axisymmetric bodies with porous walls and constant temperature or heat flux. Lien and Chen (1987) used the Mangler transformation and potential outer flow velocity, while Wang and Kleinstreuer (1988) introduced a new coordinate transformation to reduce the streamwise dependence in the coupled boundary layer equations. More recently, Wang (1993) studied numerically the steady mixed convection micropolar boundary layer past two-dimensional or axisymmetric bodies with permeable walls. However, these authors have introduced three material parameters.

Motivated by the above studies, and also those by Chen and Mucoglu (1977), Mucoglu and Chen (1978) and Nazar *et al.* (2002e) for similar problems in viscous fluids, this present study analyze the mixed convection boundary layer flow about a sphere in a micropolar fluid subject to constant temperature and constant heat flux. However, only two parameters are introduced in this study, which are the material parameter and the mixed convection parameter.

1.7.5 The Keller-box Method

An efficient and accurate implicit numerical scheme was devised for parabolic partial differential equations by Keller in 1970 (Keller, 1970,1971), namely the Keller-box method. This method has several very desirable features that make it appropriate for the solution of all parabolic partial differential equations. The procedure includes an implicit finite difference scheme in conjunction with Newton's method for linearization.

This method has been widely used and it seems to be the most flexible of the common methods. It has been tested extensively on laminar boundary layer flows (Keller and Cebeci, 1971) and turbulent boundary layer flows (Keller and Cebeci, 1972; Cebeci and Smith, 1974). It has also been shown by Keller and Cebeci (1972) and Mucoglu and Chen (1978) to be much faster, easier to program, more efficient and more flexible to use.

The scheme is also applicable to various types of boundary layer flow problems, which are the free and mixed convection flows. Na (1979) discussed the isothermal free convection over a vertical flat plate using the Keller-box method. Kumari *et al.* (1987) considered the mixed convection boundary layer flow over a sphere embedded in a saturated porous medium using this numerical method. Recently, Kumari *et al.* (1996) extended this method to solve the unsteady free convection flow over a continuous moving vertical surface. More recently, Pop and Na (1999) studied the effects of wavy surfaces on natural convection over a vertical frustum of a cone using this method. Later, Yih (2000) employed this method for studying the effect of uniform blowing/suction on MHD-free convection over a horizontal cylinder, and in the same year, Rees and Pop (2000) used the same method to study the effect of g-jitter on vertical free convection boundary layer flow in porous media. The application of the method to the problem of mixed convection boundary layer has been reported by Chen and Mucoglu (1977) as well as Mucoglu and Chen (1978), for mixed convection problems about a sphere with constant temperature and constant heat flux, respectively.

On the other hand, Rees and Bassom (1996) used the Keller-box method to study the Blasius boundary layer flow of a micropolar fluid. This implicit method is also employed in other problems involving micropolar fluid such as those by Bhattacharyya and Pop (1996), and Rees and Pop (1998) who considered problems of free convection boundary layer from cylinder of elliptic cross-section and vertical flat plate, respectively.

As for this present study, in order to verify that the Keller-box method is an appropriate method to be applied for solving the governing equations of the present problems, we have first solved the problem of free convection boundary layer over a nonisothermal vertical flat plate in a viscous fluid using this numerical method (Roslinda Nazar and Norsarahaida Amin, 2003a). The numerical results show very good agreement with other methods such as the local nonsimilarity (Kao, 1976), strained coordinate (Kao *et al.*, 1977) and Merk-type series (Yang *et al.*, 1982). We found that the Keller-box method is simpler and accurate, and can easily be extended to problems in micropolar fluids. Therefore, in this present study, we use this particular implicit finite-difference scheme for all the boundary layer problems in micropolar fluids.

1.8 Governing Equations

In this section, we derive the governing equations for free and mixed convection boundary layer flows over horizontal circular cylinders in micropolar fluids. The approximations and transformations employed in the analysis of these flow problems are outlined in the next Subsections 1.8.1 through 1.8.4, namely the Boussinesq approximation, the non-dimensional transformation, the boundary layer approximation and the non-similar transformation, respectively.

1.8.1 The Dimensional Equations and Boussinesq Approximation

The complete, dimensional form of the continuity, momentum and thermal energy equations for a viscous and incompressible fluid of steady flow, simplified only to the extent that we assume all the fluid properties, except the density, are constant and neglect the viscous dissipation effects, are given in vectorial form by (see Gebhart *et al.* (1988) or Pop and Ingham (2001)),

$$\nabla \cdot \overline{V} = 0 \tag{1.8.1}$$

$$(\overline{V} \cdot \nabla)\overline{V} = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{V} + \frac{\rho - \rho_\infty}{\rho_\infty}\overline{g} \tag{1.8.2}$$

$$(\overline{V} \cdot \nabla)\overline{T} = \alpha\ \nabla^2 \overline{T} \tag{1.8.3}$$

and for an incompressible micropolar fluid, we have

$$\nabla \cdot \overline{V} = 0 \tag{1.8.4}$$

$$(\overline{V} \cdot \nabla)\overline{V} = -\frac{1}{\rho}\nabla \overline{p} + \frac{(\mu + \kappa)}{\rho}\nabla^2 \overline{V} + \frac{\rho - \rho_\infty}{\rho_\infty}\overline{g} + \kappa\left(\frac{\partial \overline{N}}{\partial \overline{y}}\boldsymbol{i} - \frac{\partial \overline{N}}{\partial \overline{x}}\boldsymbol{j}\right) \tag{1.8.5}$$

$$(\overline{V} \cdot \nabla)\overline{T} = \alpha\ \nabla^2 \overline{T} \tag{1.8.6}$$

$$(\overline{V} \cdot \nabla)\overline{N} = \frac{\gamma}{\rho J}\nabla^2 \overline{N} + \frac{\kappa}{\rho J}\left(-2\overline{N} + \frac{\partial \overline{v}}{\partial \overline{x}} - \frac{\partial \overline{u}}{\partial \overline{y}}\right) \tag{1.8.7}$$

with an additional term and parameters in the momentum equation (1.8.5) and the new angular momentum equation (1.8.7), for micropolar fluid. $\overline{V}$ is the velocity vector, $\overline{T}$ is the fluid temperature, $\overline{N}$ is the microrotation component normal to the $(\overline{x}, \overline{y})$ - plane, $\overline{p}$ is the pressure, $\overline{g}$ is the gravitation acceleration vector, μ is the dynamic viscosity, ν is the kinematic viscosity, ρ is the fluid density, ρ_∞ is the constant local density, $\alpha = \frac{\nu}{\Pr}$ is the thermal diffusivity with Pr is the Prandtl number, $\overline{\nabla}^2$ is the Laplacian operator, κ is the vortex viscosity, J is the microinertia density, γ is the spin gradient viscosity, $\overline{u}$ and $\overline{v}$ are the velocity components along $\overline{x}$ and $\overline{y}$ directions, respectively, and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors.

For many actual fluids and flow conditions, a simple and convenient way to express the density difference (ρ - ρ_∞) in the buoyancy term of the momentum equations (1.8.2) and (1.8.5) is given by the Boussinesq approximation reported by Pop and Ingham (2001)

$$\rho = \rho_\infty \left[1 - \beta\left(\overline{T} - T_\infty\right)\right] \qquad (1.8.8a)$$

Here β is the thermal expansion coefficient and T_∞ is the temperature of the ambient medium. If the density ρ varies linearly with $\overline{T}$ over the range of values of the physical quantities encountered in the transport process, β in equation (1.8.8a) is simply

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \overline{T}}\right)_{\overline{p}} . \qquad (1.8.8b)$$

Equation (1.8.8a) is a good approximation for the variation of the density, and it is known as the Boussinesq approximation (Gebhart *et al.*,1988; Bejan, 1984), which states that: (1) all variations in fluid properties can be completely ignored except for density in momentum equation, and (2) the density is considered to vary with temperature only, and its variations can be ignored everywhere except where they give rise to buoyancy force. The Boussinesq approximation is discussed in detail by Tritton (1985).

In the following chapters, namely Chapters 3 to 6, we discuss problems in viscous and incompressible micropolar fluids, therefore, we give here in this introductory chapter the governing equations for problems in this type of fluid. Even though there are two main types of geometry that we consider in this present study as mentioned previously, in this introductory chapter we only discuss problems over horizontal circular cylinder. In addition, both free and mixed convection problems are considered, with two types of boundary conditions, i.e. constant surface temperature and constant surface heat flux.

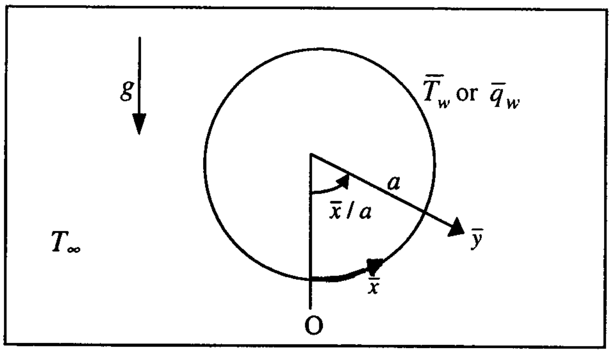

Figure 1.1 Physical model and coordinate system for free convection from a horizontal circular cylinder

For a problem of free convection, consider a heated horizontal circular cylinder of radius a, which is immersed in a viscous and incompressible micropolar fluid of ambient temperature T_∞ and a constant surface temperature $\overline{T}_w$ (or heated with a constant surface heat flux $\overline{q}_w$) as shown in Figure 1.1. It is assumed that the surface temperature of the cylinder is $\overline{T}_w$, where $\overline{T}_w > T_\infty$ (or $\overline{q}_w > 0$).

On the other hand, for a problem of mixed convection, consider a horizontal circular cylinder of radius a, which is heated to a constant temperature $\overline{T}_w$ (or heated with a constant surface heat flux $\overline{q}_w$) and is immersed in a viscous and incompressible micropolar fluid of uniform free stream U_∞ and ambient temperature T_∞, as shown in Figure 1.2. The orthogonal coordinates $\overline{x}$ and $\overline{y}$ are measured along the surface of the cylinder, starting with the lower stagnation point, and normal to it, respectively.

The fluid properties are assumed to be constant except for variation in density, which induce the buoyancy force. Using Boussinesq approximation (1.8.8a), the governing dimensional equations (1.8.4) – (1.8.7) can be written in Cartesian coordinate system as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0 \qquad (1.8.9)$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{(\mu+\kappa)}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)+\kappa\frac{\partial \bar{N}}{\partial \bar{y}}+g\beta(\bar{T}-T_\infty)\sin\left(\frac{\bar{x}}{a}\right) \qquad (1.8.10)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}}+\frac{(\mu+\kappa)}{\rho}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right)-\kappa\frac{\partial \bar{N}}{\partial \bar{x}}-g\beta(\bar{T}-T_\infty)\cos\left(\frac{\bar{x}}{a}\right) \qquad (1.8.11)$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}=\alpha\left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right) \qquad (1.8.12)$$

$$\bar{u}\frac{\partial \bar{N}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{N}}{\partial \bar{y}}=-\frac{2\kappa}{\rho J}\bar{N}+\frac{\kappa}{\rho J}\left(\frac{\partial \bar{v}}{\partial \bar{x}}-\frac{\partial \bar{u}}{\partial \bar{y}}\right)+\frac{\gamma}{\rho J}\left(\frac{\partial^2 \bar{N}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{N}}{\partial \bar{y}^2}\right) \qquad (1.8.13)$$

We assume γ is constant and is given by

$$\gamma=\left(\mu+\frac{\kappa}{2}\right)J \qquad (1.8.14)$$

and this is invoked in order to allow the field equations to predict the correct behaviour in the limiting case when microstructure effects become negligible, and the microrotation, $\bar{N}$, reduces to the angular velocity (see Ahmadi (1976)).

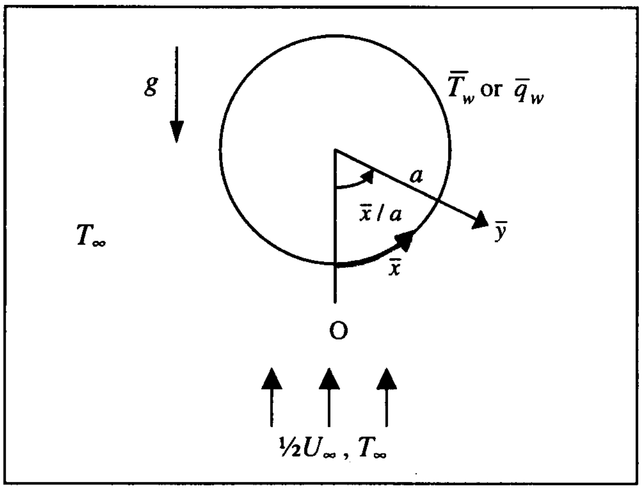

Figure 1.2 Physical model and coordinate system for mixed convection from a horizontal circular cylinder

The corresponding dimensional boundary conditions appropriate to equations (1.8.9) – (1.8.13) are as follows:

Free and mixed convection:

$\bar{y} = 0$: $\quad \bar{u} = \bar{v} = 0, \quad \bar{N} = -n\dfrac{\partial \bar{u}}{\partial \bar{y}}$ (1.8.15a)

(i) $\bar{T} = \bar{T}_w$ for constant temperature (1.8.15b)

(ii) $\dfrac{\partial \bar{T}}{\partial \bar{y}} = -\dfrac{\bar{q}_w}{k}$ for constant heat flux (1.8.15c)

Free convection:

$\bar{y} \to \infty$: $\quad \bar{u} \to 0, \quad \bar{v} \to 0, \quad \bar{N} \to 0, \quad \bar{p} \to p_\infty, \quad \bar{T} \to T_\infty$ (1.8.16a)

Mixed convection:

$\bar{y} \to \infty$: $\quad \bar{u} \to \bar{u}_e(\bar{x}), \quad \bar{v} \to 0, \quad \bar{N} \to 0, \quad \bar{p} \to p_\infty, \quad \bar{T} \to T_\infty$ (1.8.16b)

where n is a constant and $0 \le n \le 1$, and that a uniform stream $(1/2)U_\infty$ is flowing vertically upwards over the cylinder, so that the free stream velocity $\bar{u}_e(\bar{x})$ for the boundary layer equations is $\bar{u}_e(\bar{x}) = U_\infty \sin(\bar{x}/a)$ (Merkin, 1977a). The value $n = 0$, which indicates $\bar{N}(\bar{x}, 0) = 0$, represents concentrated particle flows in which the particle density is sufficiently great that microelements close to the wall are unable to rotate (Jena and Mathur, 1981). This condition is also called "strong" interaction (Guram and Smith, 1980). The case corresponding to $n = ½$ results in the vanishing of antisymmetric part of the stress tensor and represents weak concentration (Ahmadi, 1976). In this case, the particle rotation is equal to fluid vorticity at the boundary for fine particle suspension. When $n = 1$, we have flows which are representative of turbulent boundary layers (Peddieson, 1972). The case of $n = ½$ is considered in the present study.

1.8.2 The Non-Dimensional Equations

In order to solve the dimensional governing equations (1.8.9) – (1.8.13) subject to the boundary conditions (1.8.15) and (1.8.16), we will non-dimensionalize those equations by introducing the following non-dimensional variables:

For free convection:

i) Constant temperature

$$\hat{x} = \bar{x}/a, \quad \hat{y} = \mathrm{Gr}^{1/4}(\bar{y}/a), \quad \hat{u} = (a/\nu)\mathrm{Gr}^{-1/2}\,\bar{u}, \quad \hat{v} = (a/\nu)\mathrm{Gr}^{-1/4}\bar{v}, \quad (1.8.17a)$$

$$\hat{p} = \frac{\bar{p} - p_\infty}{\mathrm{Gr}(\nu^2/a^2)\rho}, \quad \hat{T} = \frac{\bar{T} - T_\infty}{\bar{T}_w - T_\infty}, \quad \hat{N} = (a^2/\nu)\mathrm{Gr}^{-3/4}\,\bar{N}, \quad (1.8.17a)$$

where Gr = $g\beta(\bar{T}_w - T_\infty)a^3/\nu^2$ is the Grashof number, and the microinertia density J is taken to be $J = a^2\mathrm{Gr}^{-1/2}$.

ii) Constant heat flux

$$\hat{x} = \bar{x}/a, \quad \hat{y} = \mathrm{Gr}^{1/5}(\bar{y}/a), \quad \hat{u} = (a/\nu)\mathrm{Gr}^{-2/5}\,\bar{u}, \quad \hat{v} = (a/\nu)\mathrm{Gr}^{-1/5}\bar{v}, \quad (1.8.17b)$$

$$\hat{p} = \frac{\bar{p} - p_\infty}{\mathrm{Gr}(\nu^2/a^2)\rho}, \quad \hat{T} = \mathrm{Gr}^{1/5}\left(\frac{\bar{T} - T_\infty}{a\bar{q}_w/k}\right), \quad \hat{N} = (a^2/\nu)\mathrm{Gr}^{-3/5}\,\bar{N}, \quad (1.8.17b)$$

where Gr = $g\beta(a\bar{q}_w/k)\,a^3/\nu^2$ and the microinertia density is now, $J = a^2\mathrm{Gr}^{-2/5}$.

For mixed convection:

i) Constant temperature

$$\hat{x} = \bar{x}/a, \quad \hat{y} = \mathrm{Re}^{1/2}(\bar{y}/a), \quad \hat{u} = \bar{u}/U_\infty, \quad \hat{v} = \mathrm{Re}^{1/2}\bar{v}/U_\infty, \quad (1.8.17c)$$

$$\hat{u}_e(\hat{x}) = \frac{\bar{u}_e(\bar{x})}{U_\infty}, \quad \hat{p} = \frac{\bar{p} - p_\infty}{\rho U_\infty^2}, \hat{T} = \frac{\bar{T} - T_\infty}{\bar{T}_w - T_\infty}, \quad \hat{N} = \mathrm{Re}^{-1/2}(a/U_\infty)\,\bar{N}, \quad (1.8.17c)$$

ii) Constant heat flux

$$\hat{x} = \bar{x}/a, \quad \hat{y} = \mathrm{Re}^{2/5}(\bar{y}/a), \quad \hat{u} = \bar{u}/U_\infty, \quad \hat{v} = \mathrm{Re}^{2/5}\bar{v}/U_\infty, \tag{1.8.17d}$$

$$\hat{u}_e(\hat{x}) = \frac{\bar{u}_e(\bar{x})}{U_\infty}, \ \hat{p} = \frac{\bar{p} - p_\infty}{\rho U_\infty^2}, \ \hat{T} = \mathrm{Re}^{2/5}\left(\frac{\bar{T} - T_\infty}{a\bar{q}_w/k}\right), \quad \hat{N} = \mathrm{Re}^{-2/5}(a/U_\infty)\,\bar{N} \tag{1.8.17d}$$

where $\mathrm{Re} = U_\infty a/\nu$ is the Reynolds number. The microinertia density J is taken to be $J = a\nu/U_\infty$.

On using the expressions (1.8.17) in equations (1.8.9) – (1.8.13), we obtain

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{1.8.18}$$

For free convection:

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{(1+\mathrm{K})}{\mathrm{Gr}^{1/2}}\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right) + \frac{\mathrm{K}}{\mathrm{Gr}^{1/2}}\frac{\partial \hat{N}}{\partial \hat{y}} + \hat{T}\sin\hat{x} \tag{1.8.19a}$$

$$\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{(1+\mathrm{K})}{\mathrm{Gr}^{1/2}}\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right) - \frac{\mathrm{K}}{\mathrm{Gr}^{1/2}}\frac{\partial \hat{N}}{\partial \hat{x}} - \hat{T}\cos\hat{x} \tag{1.8.20a}$$

$$\hat{u}\frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}}{\partial \hat{y}} = \frac{1}{\mathrm{Pr}\,\mathrm{Gr}^{1/2}}\left(\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2}\right) \tag{1.8.21a}$$

$$\hat{u}\frac{\partial \hat{N}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{N}}{\partial \hat{y}} = -2\mathrm{K}\,\hat{N} + \mathrm{K}\left(\frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\partial \hat{u}}{\partial \hat{y}}\right) + \frac{(1+\frac{\mathrm{K}}{2})}{\mathrm{Gr}^{1/2}}\left(\frac{\partial^2 \hat{N}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{N}}{\partial \hat{y}^2}\right) \tag{1.8.22a}$$

For mixed convection:

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{(1+\mathrm{K})}{\mathrm{Re}}\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right) + \frac{\mathrm{K}}{\mathrm{Re}}\frac{\partial \hat{N}}{\partial \hat{y}} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\hat{T}\sin\hat{x} \tag{1.8.19b}$$

$$\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{(1+\mathrm{K})}{\mathrm{Re}}\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right) - \frac{\mathrm{K}}{\mathrm{Re}}\frac{\partial \hat{N}}{\partial \hat{x}} - \frac{\mathrm{Gr}}{\mathrm{Re}^2}\hat{T}\cos\hat{x} \tag{1.8.20b}$$

$$\hat{u}\frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}}{\partial \hat{y}} = \frac{1}{\mathrm{Pr}\,\mathrm{Re}}\left(\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2}\right) \tag{1.8.21b}$$

$$\hat{u}\frac{\partial \hat{N}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{N}}{\partial \hat{y}}=-2\mathrm{K}\hat{N}+\mathrm{K}\left(\frac{\partial \hat{v}}{\partial \hat{x}}-\frac{\partial \hat{u}}{\partial \hat{y}}\right)+\frac{(1+\frac{\mathrm{K}}{2})}{\mathrm{Re}}\left(\frac{\partial^2 \hat{N}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{N}}{\partial \hat{y}^2}\right) \qquad (1.8.22b)$$

where $\mathrm{K}=\frac{\kappa}{\mu}$ is the micropolar or material parameter. Non-zero values of K cause coupling between the fluid flow and the microrotation component $\hat{N}$. The boundary conditions (1.8.15) and (1.8.16) now become (in non-dimensional form):

Free and mixed convection:

$$\hat{y}=0: \qquad \hat{u}=\hat{v}=0, \qquad \hat{N}=-\frac{1}{2}\frac{\partial \hat{u}}{\partial \hat{y}} \qquad (1.8.23a)$$

(i) $\hat{T}=1$ for constant temperature (1.8.23b)

(ii) $\frac{\partial \hat{T}}{\partial \hat{y}}=-1$ for constant heat flux (1.8.23c)

Free convection:

$$\hat{y}\to\infty: \qquad \hat{u}\to 0,\quad \hat{v}\to 0,\quad \hat{N}\to 0,\quad \hat{p}\to 0,\quad \hat{T}\to 0 \qquad (1.8.24a)$$

Mixed convection:

$$\hat{y}\to\infty: \qquad \hat{u}\to \hat{u}_e(\hat{x}),\quad \hat{v}\to 0,\quad \hat{N}\to 0,\quad \hat{p}\to 0,\quad \hat{T}\to 0 \qquad (1.8.24b)$$

1.8.3 Boundary Layer Approximation

In this section, we present the simplified boundary layer approximations, while the detailed descriptions of the basic derivations of the boundary layer equations can be found in any elementary fluid dynamics and heat transfer books such as Batchelor (1967), Chapman (1980) and Acheson (1990).

We now invoke the boundary layer approximation for free convection problem, by introducing the following scaling (for the case of constant temperature):

$$\hat{x} = x,\quad \hat{y} = \mathrm{Gr}^{-1/4}y,\quad \hat{u} = u,\quad \hat{v} = \mathrm{Gr}^{-1/4}v,\quad \hat{p} = p,\quad \hat{N} = \mathrm{Gr}^{1/4}N,\quad \hat{T} = \theta \qquad (1.8.25a)$$

and for mixed convection problem:

$$\hat{x} = x,\quad \hat{y} = \mathrm{Re}^{-1/2}y,\quad \hat{u} = u,\quad \hat{v} = \mathrm{Re}^{-1/2}v,\quad \hat{p} = p,\quad \hat{N} = \mathrm{Re}^{1/4}N,\quad \hat{T} = \theta,\quad \hat{u}_e = u_e \qquad (1.8.25b)$$

Substituting (1.8.25a) into equations (1.8.18), (1.8.19a) - (1.8.22a) and formally letting $\mathrm{Gr} \to \infty$ ($\mathrm{Gr} >> 1$), while substituting (1.8.25b) into equations (1.8.18), (1.8.19b) - (1.8.22b) and formally letting $\mathrm{Re} \to \infty$ ($\mathrm{Re} >> 1$), leads to the following boundary layer equations:

Free and mixed convection:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.8.26)$$

Free convection:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (1+\mathrm{K})\frac{\partial^2 u}{\partial y^2} + \mathrm{K}\frac{\partial N}{\partial y} + \theta \sin x \qquad (1.8.27a)$$

Mixed convection:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (1+\mathrm{K})\frac{\partial^2 u}{\partial y^2} + \mathrm{K}\frac{\partial N}{\partial y} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\theta \sin x \qquad (1.8.27b)$$

Free and mixed convection:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial y^2} \qquad (1.8.28)$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = -2\mathrm{K}N - \mathrm{K}\frac{\partial u}{\partial y} + \left(1+\frac{\mathrm{K}}{2}\right)\frac{\partial^2 N}{\partial y^2} \qquad (1.8.29)$$

$$\frac{\partial p}{\partial y} = 0,\qquad p = p(x) \qquad (1.8.30)$$

As $\mathrm{Gr} \to \infty$ and $\mathrm{Re} \to \infty$, we have

$$\frac{\partial p}{\partial y} = O(\mathrm{Gr}^{-1/4}) = 0 \quad \text{and} \quad \frac{\partial p}{\partial x} = O(\mathrm{Gr}^{-1/4}) = 0 \text{ (free convection)} \qquad (1.8.31a,b)$$

$$\frac{\partial p}{\partial y} = O(\mathrm{Re}^{-1/2}) = 0, \text{ (mixed convection)} \tag{1.8.31c}$$

and from Bernoulli's equation (see Cebeci and Bradshaw (1988)), we get

$$-\frac{\partial p}{\partial x} = u_e \frac{du_e}{dx} \quad \text{(mixed convection)} \tag{1.8.31d}$$

which will be used in equation (1.8.27b).

The boundary conditions (1.8.23) and (1.8.24) become:

Free and mixed convection:

$$y = 0: \qquad u = v = 0, \qquad N = -\frac{1}{2}\frac{\partial u}{\partial y} \tag{1.8.32a}$$

$$\text{(i)} \qquad \theta = 1 \qquad \text{for constant temperature} \tag{1.8.32b}$$

$$\text{(ii)} \qquad \frac{\partial \theta}{\partial y} = -1 \qquad \text{for constant heat flux} \tag{1.8.32c}$$

Free convection:

$$y \to \infty: \qquad u \to 0, \quad N \to 0, \quad \theta \to 0 \tag{1.8.33a}$$

Mixed convection:

$$y \to \infty: \qquad u \to u_e(x), \quad N \to 0, \quad \theta \to 0 \tag{1.8.33b}$$

1.8.4 Nonsimilar Transformation

To solve equations (1.8.26) - (1.8.29) subject to the boundary conditions (1.8.32) and (1.8.33), we assume the following variables:

$$\psi = x\,F(x,y), \qquad \theta = \theta\,(x,y), \quad N = x\,G(x,y) \tag{1.8.34}$$

where ψ is the non-dimensional stream function defined in the usual way as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{1.8.35}$$

and therefore, equation (1.8.26) is satisfied automatically. Substituting equations (1.8.34) and (1.8.35) into equations (1.8.27) – (1.8.29), we get the following transformed equations:

Free convection:

$$(1+\mathrm{K})\frac{\partial^3 F}{\partial y^3}+F\frac{\partial^2 F}{\partial y^2}-\left(\frac{\partial F}{\partial y}\right)^2+\frac{\sin x}{x}\theta+\mathrm{K}\frac{\partial G}{\partial y}=x\left(\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x\partial y}-\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial y^2}\right) \quad (1.8.36)$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}+F\frac{\partial\theta}{\partial y}=x\left(\frac{\partial F}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial\theta}{\partial y}\right) \quad (1.8.37)$$

$$\left(1+\frac{\mathrm{K}}{2}\right)\frac{\partial^2 G}{\partial y^2}+F\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}G-\mathrm{K}\left(2G+\frac{\partial^2 F}{\partial y^2}\right)=x\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}\right) \quad (1.8.38)$$

subject to the boundary conditions

$$F=\frac{\partial F}{\partial y}=0,\quad \theta=1 \text{ or } \frac{\partial\theta}{\partial y}=-1,\quad G=-\frac{1}{2}\frac{\partial^2 F}{\partial y^2}\quad \text{on } y=0, \quad (1.8.39a)$$

$$\frac{\partial F}{\partial y}\to 0,\quad \theta\to 0,\quad G\to 0\quad \text{as } y\to\infty, \quad (1.8.39b)$$

Mixed convection:

$$(1+\mathrm{K})\frac{\partial^3 F}{\partial y^3}+F\frac{\partial^2 F}{\partial y^2}-\left(\frac{\partial F}{\partial y}\right)^2+\frac{\sin x\cos x}{x}+\lambda\frac{\sin x}{x}\theta+\mathrm{K}\frac{\partial G}{\partial y}$$
$$=x\left(\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x\partial y}-\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial y^2}\right) \quad (1.8.40)$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}+F\frac{\partial\theta}{\partial y}=x\left(\frac{\partial F}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial\theta}{\partial y}\right) \quad (1.8.41)$$

$$\left(1+\frac{\mathrm{K}}{2}\right)\frac{\partial^2 G}{\partial y^2}+F\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}G-\mathrm{K}\left(2G+\frac{\partial^2 F}{\partial y^2}\right)=x\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}\right) \quad (1.8.42)$$

subject to the boundary conditions

$$F=\frac{\partial F}{\partial y}=0,\quad \theta=1 \text{ or } \frac{\partial\theta}{\partial y}=-1,\quad G=-\frac{1}{2}\frac{\partial^2 F}{\partial y^2}\quad \text{on } y=0 \quad (1.8.43a)$$

$$\frac{\partial F}{\partial y}\to\frac{\sin x}{x},\quad \theta\to 0,\quad G\to 0\quad \text{as } y\to\infty \quad (1.8.43b)$$

where λ is the mixed convection parameter which can be written as

$\lambda = \dfrac{\mathrm{Gr}}{\mathrm{Re}^2}$ (for constant temperature) and $\lambda = \dfrac{\mathrm{Gr}}{\mathrm{Re}^{5/2}}$ (for constant heat flux).

The physical quantities of principal interest in this study are the heat transfer coefficient (case of constant temperature) and the skin friction coefficient, and they can be written as (an example given for the case of free convection boundary layer on an isothermal horizontal circular cylinder in a micropolar fluid)

$$Q_w = \frac{a\,\mathrm{Gr}^{-1/4}}{k(\bar{T}_w - T_\infty)} q_w, \qquad C_f = \frac{\mathrm{Gr}^{-3/4} a^2}{\mu \nu} \tau_w, \tag{1.8.44}$$

where

$$q_w = -k\left(\frac{\partial \bar{T}}{\partial \bar{y}}\right)_{\bar{y}=0}, \qquad \tau_w = \left(\mu + \frac{\kappa}{2}\right)\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} . \tag{1.8.45}$$

Using the non-dimensional variables (1.8.17a), boundary layer approximations (1.8.25a), variables (1.8.34) and the boundary condition (1.8.39a) for G, the non-dimensional heat transfer and skin friction coefficients are

$$Q_w = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0}, \qquad C_f = \left(1 + \frac{\mathrm{K}}{2}\right) x \left(\frac{\partial^2 F}{\partial y^2}\right)_{y=0}, \tag{1.8.46}$$

respectively, and the non-dimensional wall temperature distribution (for the case of constant heat flux) is given by

$$\theta(x,0) = \theta_w(x) \tag{1.8.47}$$

The non-dimensional forms of these physical quantities are similar for all of the problems in micropolar fluids considered in this thesis.

The heat transfer (or wall temperature) and skin friction coefficients are important because in any practical situations, we have to know how strong a body is heated; the rate of heat transfer from the body to the fluid; what kind of materials we have to use in order to avoid the body to be exposed to high temperatures, etc. All the practical devices that operate with the use of heat are designed based on theoretical or experimental heat transfer characteristics. For example, nuclear devices, insulation of buildings, etc.

The skin friction coefficient on the other hand, is also important for practical problems because it determines the heating of the body due to shear stress on the body and the heat loss by friction. From the mathematical point of view (or theoretically) this coefficient is important, for example, to determine the nature of the flow and its separation (when the skin friction equals zero) from the body surface. When the flow separates from the body, it becomes unstable and rotational. The phenomenon of separation is very important for designing and building of a plane. In short, there are many such kinds of practical situations where the skin friction coefficient plays a fundamental importance.

The governing equations discussed in this section are for free and mixed convection boundary layer flows over horizontal circular cylinders in micropolar fluids. These problems are discussed further in Chapters 3 and 5, while problems about spheres are presented in Chapters 4 and 6. The resulting nonsimilar boundary layer equations for the specific problem of free convection boundary layer flow over an isothermal horizontal circular cylinder in a micropolar fluid discussed in this section will be used to describe in detail the stepwise development of the Keller-box method, which is presented in the next chapter, Chapter 2.