P-wave Diffusion in Fluid-Saturated Medium

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Abstract This paper considers the propagating P-waves in the fluid-saturated mediums that are categorized to fall into two distinct groups: insoluble and soluble mediums. P-waves are introduced with slowness in accordance to Snell Law and are shown to relate to the medium displacement and wave diffusion. Consequently, the results bear out that the propagating P-waves in the soluble medium share similar diffusive characteristic as of insoluble medium. Nonetheless, our study on fluid density in the mediums show that high density fluid promotes diffusive characteristic whiles low density fluid endorses non-diffusive P-wave.

Keywords P-wave; fluid-saturated mediums; medium displacement; wave diffusion.

1   Introduction

P-waves are referred to the primary waves when the phenomenon of an earthquake is being discussed. Engineers have studied widely and estimated the wet-rock P-wave velocity from the dry-rock P-wave velocity (refer e.g. Kahraman [1]). The dependence of P- and S-wave attenuation on strain amplitude and frequency had also been studied experimentally in dry and water-saturated sandstone samples under a confining pressure (e.g. Mashinkii [2]).

The protection of underground structures against dynamic loadings has long been a topic of great interest in defense engineering. Hence, the P-wave propagation and attenuation in rock shelter layer with an inclusion or filled medium had been studied in relation to such engineering interest (e.g. Wang et al. [3]). Furthermore, the problem of diffraction of waves due to plane harmonic P-wave incident normally on a line crack situated in an infinite micropolar elastic medium was studied as well in Midya et al. [4]. Nevertheless, Weihua et al. [12] has recently obtained an analytical solution is obtained for two-dimensional scattering and diffraction of plane P-waves by circular-arc alluvial valleys with shallow saturated soil deposits. We were taken aback that similar research is taken place in medical cardiology (e.g. Materazzo [13]) where P-waves are used to monitor the risk assessment of the patient after undergoing surgery.

We organized this paper as such that section 1 is our brief review of current situation, section 2 is the problem formulation and mathematical formulations, section 3 is our analytical discussions on our results and finally section 4 posted our concluding remarks.

2   Problem Formulation

The governing equations for the displacement is given by

$$\mu \nabla^2 u + (\lambda + \mu) \nabla (\nabla u) + \rho F = \rho \frac{\partial^2 u}{\partial t^2}. \tag{1}$$

Boundary conditions for $z = 0$ and the stresses are

$$\sigma_z \equiv (\lambda + \mu) \frac{\partial u_z}{\partial z} + \lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0,$$

$$\tau_{xz} \equiv \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0,$$

$$\tau_{yz} \equiv \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) = 0.$$

Initial conditions for $t = 0$ are

$$u(x, y, z) = 0,$$

$$\frac{\partial u_x}{\partial t} = \frac{\partial u_y}{\partial t} = \frac{\partial u_z}{\partial t} = 0.$$

In linear elasticity for isotropic medium, λ and μ denote the Lame parameters for the stress σ_z, τ_{xz}, τ_{yz} and the displacements u_x, u_y and u_z are continuous everywhere. F is the body force in the direction of x, y, z respectively and ρ is the density.

Provided that ω is the angular velocity or frequency, k is the wavenumber, c is the wave velocity along with the dispersion relation $\omega = ck$, the group velocity of a wave is the velocity with the overall shape of the wave's amplitudes which is also known as envelope of the waves. Several waves add together to form a single wave shape (called the envelope). In other words, the phase velocity is the average velocity of the components, given by $V_P = \omega/k$. The group velocity is velocity of the envelope, given by $V_g = d\omega/dk$ [9].

When this is applied to the problem of P-wave's propagation in fluid saturated medium, the group velocity reflects the apparent velocity of the surface displacement or the overall shape of the P- wave's amplitude at the boundary of the medium. The envelope is formed by the grouping of P-waves.

In this research, the apparent velocity V_{app} or surface displacement velocity is measured along the boundary of the similar density medium in accordance to Snell law:

$$\frac{V_{app}}{\sin f} = \frac{c}{\sin e}, \tag{2}$$

where e is the incident angle and f is the refraction angle made by the P- and S-waves. However, this research aims to study P-waves only. For the case of fluid-saturated medium, there exists variation in density within medium [7, 8]. There exists slowness for fluid-saturated medium [5, 6] i.e. the apparent velocity of the displacement at the boundary is slower than phase velocity of the wave in the medium,

$$V_{app} < c. \tag{3}$$

In this paper, the elastic wave equation will be solved by extracting P-wave's displacements in the different types of medium that satisfied (2) and (3). By using the divergence operator,

$$\nabla^2 u = \nabla \left(\nabla u \right) - \nabla \times \left(\nabla \times u \right),$$

equation (1) is reduced to

$$\alpha^2 \nabla \left(\nabla \cdot u \right) - \beta^2 \nabla \times \nabla \times u + F = \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where α and β are given as $\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \beta = \sqrt{\frac{\mu}{\rho}}$. Here, α and β are the velocities for the P-wave and S-wave and equation (4) is the elastic wave equation [10]. Hence, this modeling is only valid for elastic medium and it is necessary to reduce the right hand side (RHS) of (4) by letting

$$u = \exp \left[i \left(kx - ct \right) \right]. \tag{5}$$

Inserting second order derivative of (5) into the right hand side of (4), the equation reads

$$\alpha^2 \nabla \left(\nabla \cdot u \right) - \beta^2 \nabla \times \nabla \times u + F = -\omega^2 u. \tag{6}$$

Since elastic wave consists of irrotational P-wave and solenoidal by S-wave [10], the displacement can be written as

$$u = u_P + u_S. \tag{7}$$

For irrotational P-waves, the vorticity $\nabla \times u_P = 0$. Thus, the relation (7) reads

$$\nabla u_P \neq 0, \nabla \times u_P = 0, \nabla u_S = 0, \nabla \times u_S \neq 0. \tag{8}$$

Inserting (7) and (8) into (6), the equation yields

$$\alpha^2 \left(\nabla^2 u_P + k_P{}^2 u_P \right) + \beta^2 \left(\nabla^2 u_S + k_S{}^2 u_S \right) = 0, \ \ \omega_P = \frac{\alpha}{k_P}, \omega_S = \frac{\beta}{k_S}, \ \ F = 0.$$

The Helmholtz equations for P- and S-waves [11] are respectively

$$\nabla^2 u_P + k_P{}^2 u_P = 0, \ \ \nabla^2 u_S + k_S{}^2 u_S = 0.$$

Here, the Helmholtz equations are solved by utilizing Hansen vector [10] that gives

$$u_P = A \left(l a_x + n a_z \right) \exp \left[i\omega \left(t - \frac{lx + nz}{\alpha} \right) \right], \tag{9}$$

$$u_{SV} = B \left(-n a_x + l a_z \right) \exp \left[i\omega \left(t - \frac{lx + nz}{\beta} \right) \right], \tag{10}$$

$$u_{SH} = C a_y \exp \left[i\omega \left(t - \frac{lx + nz}{\beta} \right) \right], \tag{11}$$

where a_x, a_y, and a_z are the unit vectors while l and n are the vector components. Only equation (9) will be considered in this paper since the aim is to study the P-wave only. P-wave vanishes when the depth go infinity as illustrated in figure 1. Thus, the amended equation (9) or P-wave on which the amplitude reduces with depth is

$$u_P = A \left(a_x + \eta_\alpha a_z \right) \exp \left[ik \left(ct - x - \eta_\alpha z \right) \right], \ \ \eta_\alpha = \sqrt{\frac{c^2}{\alpha^2} - 1}. \tag{12}$$

η_α is always positive. The velocity of the P-wave α, is measured at the boundary of the medium or $z = 0$ and it is similar to the apparent velocity V_{app} of the surface displacement.
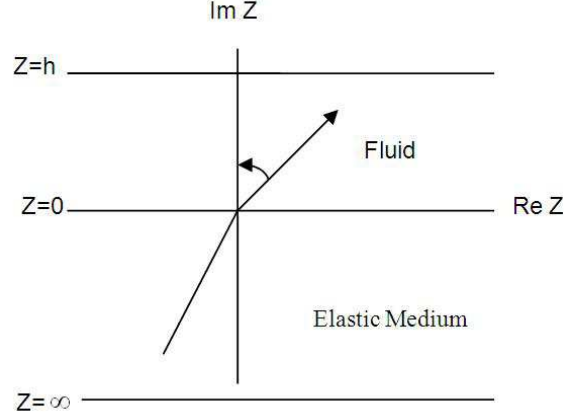
Figure 1: The P-wave's incidence and the fluid saturated medium

3   Analytical Discussions

For the displacement parallel to the propagation direction, by means of quantity η_α and amplitude A, equation (12) for x-direction of displacements with amplitude reduces with depth z gives

$$u_P = [A (a_x + \eta_\alpha a_z) \exp(-ik\eta_\alpha z)] \exp[ik(ct - x)], \quad \omega_1 = \eta_\alpha k. \tag{13}$$

Here, the quantity η_α refers to the refracted wave velocities that gives $\omega_1 = \eta_\alpha k$. From wave terminology, the term $\exp(-ik\eta_\alpha z) \exp[ik(ct - x)]$ in equation (13) potrays the harmonic waves such as diffusive with depth z if $\omega_1 = -\eta_\alpha k$ is in complex [9]. The diffusive waves are associated with attenuation of the amplitudes with the time due to certain dissipation mechanisms present in the system.

Equation (13) should be polarized to give the real quantities (frequency) equation that reads [10],

$$u_P = [A\eta_\alpha \exp(-ik\eta_\alpha z)] \exp[ik(ct - x)]. \tag{14}$$

Next, the roles of η_α will be shown. For the fluid-saturated medium, there comes the slowness induced by refraction [5, 6]. The envelope velocity at medium surface is different from the wave velocity in the medium after the slowness or $V_{app} \neq c$. For certain particular cases, the P-wave velocity in the medium is greater than the envelope's velocity that yields

$$c > \alpha, \quad \alpha = V_{app}, \tag{15}$$

Here, we propose the relation for another two types of medium conditions such that

$$c = \alpha, \quad \alpha = V_{app}, \tag{16}$$

$$c < \alpha, \quad \alpha = V_{app}. \tag{17}$$

Relations (15) and (17) are meant for the insoluble medium such that the variation of velocities c and α is significant. When the velocity c is similar to α, we presume this will

explain the soluble medium such that the fluid mixes well with the medium to give similar velocity. Yang and Tadanobu [14] and Kahraman [1] show that the high density medium promotes high wave's velocity. Hence, relation (15) will only be present when the low density fluid is saturated in the insoluble medium; the low density fluid will reshuffle the ray velocity or reduce the P-wave velocity. Eventually, relation (17) is meant for the high density fluid saturated in the insoluble medium.

The detail explanations about the tie between medium's solubility and the fluid's density will be discussed next for showing the vital roles played by the relations (15), (16) and (17) especially the displacement's characteristic.

When condition (17) is applied to the quantity η_α in equation (12), a complex solution will be obtained for η_α that reads

$$\eta_\alpha \to i\eta_\alpha. \tag{18}$$

This indicates an amendment is required for equation (14) to give (19). Hence, the P-wave's displacements for three types of mediums are

$$u_P = [Ai\eta_\alpha \exp(k\eta_\alpha z)] \exp[ik(ct - x)] \ \text{ for } c < \alpha, \tag{19}$$

$$u_P = [A\eta_\alpha \exp(-ik\eta_\alpha z)] \exp[ik(ct - x)] \ \text{ for } c = \alpha, \tag{20}$$

$$u_P = [A\eta_\alpha \exp(-ik\eta_\alpha z)] \exp[ik(ct - x)] \ \text{ for } c > \alpha. \tag{21}$$

Equations (19), (20) and (21) show that the P-waves propagation in x-direction while the diffusion is in z-direction. For a soluble medium with $c = \alpha$, equation (20) gives

$$u_P = [A\eta_\alpha] \exp[ik(ct - x)] \ \text{ for } \eta_\alpha = 0. \tag{22}$$

Indeed, the displacement does not exist for equation (20) if $\eta_\alpha = 0$. Consequently, the relation (22) implies that there is no diffraction to be done.

An insoluble medium with displacement (19) is diffusive given that the term $\omega_1 = k\eta_\alpha$ is complex [9]. Figure 2 is plotted for the diffusive P-waves with $c < \alpha$, $-5 \le x \le 5$, $k = 1$, $\alpha = 4$, $c = 2$. High density fluid promotes the diffusive characteristic for propagating P-waves in fluid saturated medium.

An insoluble medium with displacement (21) is non-diffusive given that the term $\omega_1 = k\eta_\alpha$ is real [9]. Figure 3 is plotted for the non-diffusive P-waves with $c > \alpha$, $-5 \le x \le 5$, $k = 1$, $\alpha = 2$, $c = 4$. Low density fluid promotes the non-diffusive characteristic for propagating P-waves in fluid saturated medium. However it is shown in Figure 3 that the compression exists when the polarized P-waves propagate in $x - z$ direction. The compression is related to shock wave which is formed by the wave diffraction [15].

4   Conclusion

The studies show that the P-waves in the soluble medium shares similar displacement and diffusion characteristics by insoluble medium. However, the discussion on fluid density has linked the P-waves to the diffusion. High density fluid in insoluble medium promotes diffusive P-waves while low density fluid in insoluble medium promotes non-diffusive P-waves. Consequently, the results are important for judging the possible crack done by P-waves propagation in soluble and insoluble mediums.
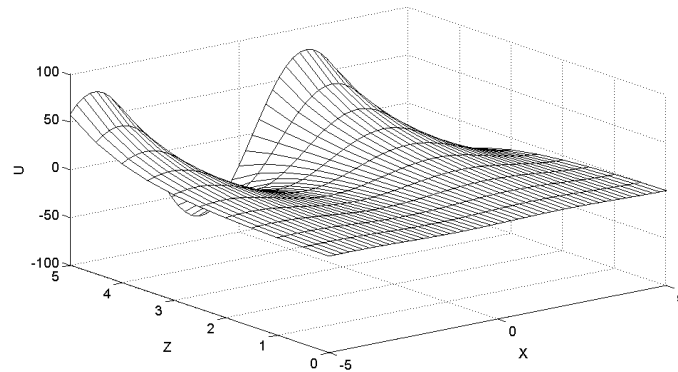
Figure 2: The diffusive characteristic by P waves propagate in the medium saturated with high density fluid. $c < \alpha$, $-5 \leq x \leq 5$, $k = 1$, $\alpha = 4$, $c = 2$
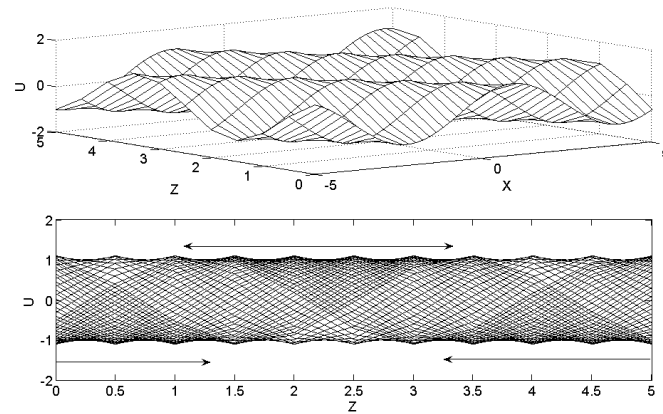
Figure 3: The non-diffusive characteristic by P waves propagate in the medium saturated with high density fluid. $c > \alpha$, $-5 \leq x \leq 5$, $k = 1$, $\alpha = 2$, $c = 4$. There exists compression of P-waves which are known as shock waves [15].

Acknowledgement

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