A NUMERICAL SOLUTION OF TRAFFIC FLOW PROBLEM FOR ONE LANE ROADWAY

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Abstract

This research is conducted to model flow of traffic on a one lane roadway by partial differential equation (PDE). Then, Finite Difference Method (FDM) is used to solve one-dimensional traffic flow equation. The Finite Difference Method involved is forward difference and central difference. In this problem, the density of cars with fixed ends is considered. The finite difference method (FDM) proceeds by replacing the derivatives in the traffic flow equations by finite difference approximations. This gives a large algebraic system of equations to be solved, which can be solved easily in mathematics software. MATLAB Distributed Computing R2010a software is used to perform the computational experiment while Microsoft Excel is used toillustrated the graphs. In this research, the effect of different step space, *h* and step time, *k* are investigated. Besides, comparison between finite difference solutions and analytical solutions will determine the accuracy of finite difference method (FDM).

Keywords: Traffic flow equation; Finite Difference Method; Macroscopic traffic flow model.

Introduction

Traffic flow is the interactions between drivers, vehicles, and infrastructure with the aim of developing an optimal road network with efficient movement of traffic and minimal traffic congestion problems. There are two classes of models in traffic flow problem which are macroscopic and microscopic. Macroscopic models concerned with average behaviour such as average speed, module area and traffic density while microscopic models concerned on individual behaviour. In this paper, the macroscopic models will be considered.

Method of characteristic is a technique for solving partial differential equations. This method is used to reduce a partial differential equation (PDE) to a family of ordinary differential equations (ODE).Meanwhile, finite difference method (FDM) is a numerical method for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. However, when FDM isused to treat numerically a PDE, the differentiable solution is approximated by somegrid function.

This research is aimed to model flow of traffic on a one lane roadway by partial different equation and solved by characteristic method and finite difference method. Purpose for modeling this model is to estimate number of cars on a one lane roadway and minimal congestion of traffic problem.Besides, we have to determine the accuracy of finite difference method.

The objectives of this research study comprises of (i)to model the flow of traffic on a one lane roadway with partial differential equation (PDE); (ii) to solve the traffic flow problem with characteristic method and finite difference method (FDM);(iii) to investigate on the effect of different step space, *h* for distance and different step time, *k* for time; and(iv) to determine the accuracy of finite difference method.The scope of study is to model flow of traffic on a one lane roadway by partial different equation. This report is carried out by taking moving cars as samples. The space distance between cars is assumed to be uniform, the density is continuous, ignored the behaviour of individual driver and each car has the same size.

Literature Review

In 1930's, the scientific study of traffic flow had begun by Greenshields (1935) with the study of models relating volume and speed with the pioneer in the use of photography relating to traffic matters and applying mathematics to traffic flow. Then, continued with the application of probability theory to the description of road traffic by Adams (1936) and the investigation of traffic performance at urban street intersections (Greenshields*et al.*, 1949) In 1950's, theoretical developments based on a variety of approaches, such as car-following, traffic wave theory which is hydrodynamic analogy and queuing theory by Wardrop (1952), Pipes (1953), Lighthill and Whitham (1955), Newell (1955), Edie and Foote (1958), and Chandler *et al.* (1958).

Traffic flow model has been modified to various models and they are solved by differential equation or partial differential equation. For example, Alkhazraji (2008) focused on modification of non-linear car following model and generally aimed at improving the validity of the model; Sohrweide*et al.* (2001) compared between four-lane arterial roadways and three-lane roadway with a centre two way left-turn lane in their work to identify which lane provided the traffic calming; Coclite*et al.* (2005) concerned with a fluid dynamic model of heavy traffic flow on a road. Precisely, they consider single conservation law, defined a road network with junctions and concluded from conservation of number of cars;Doboszczak and Forstall (2013) modelled the flow of traffic on a one lane roadway by partial differential equation (PDE) with assuming the velocity is constant for all cars, density is continuous and ignoring the individual behavior and other effects such as weather, accident and construction; Sultana *et al.* (2003) considered a fluid dynamic traffic flow model appended with a closure linear velocity-density relationship which provides a first order partial differential equation (PDE). Besides that, initial boundary value problem is treated in the paper. Analytical solution of traffic flow is present as Cauchy problem while a numerical solution of traffic flow model with initial boundary value problem is performed based on a finite difference scheme.

Traffic density and car velocity are related by only one equation, conservation of car

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

Herman (1977) assumed that under all circumstances the driver's velocity is a known function of ρ, determined by $u = u(\rho)$, then the conservation of cars implies

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0.$$

Then, (Alkhazraji, 2008) stated that the partial differential equation which was formulated to mathematically model traffic flow is

$$\frac{\partial \rho}{\partial t} + \left(u + \rho \frac{du}{d\rho}\right)\frac{\partial \rho}{\partial x} = 0.$$

Methodology

This research is aimed to model flow of traffic on a one lane roadway by partial different equation. The traffic flow model is solved explicitly by characteristic method and finite difference. At the end of the research, we will find out the models flow of traffic on a one lane roadway. With this model, it will show us how to approximate the number of cars in specific distance and reduce congestion traffic flow with some conditions.

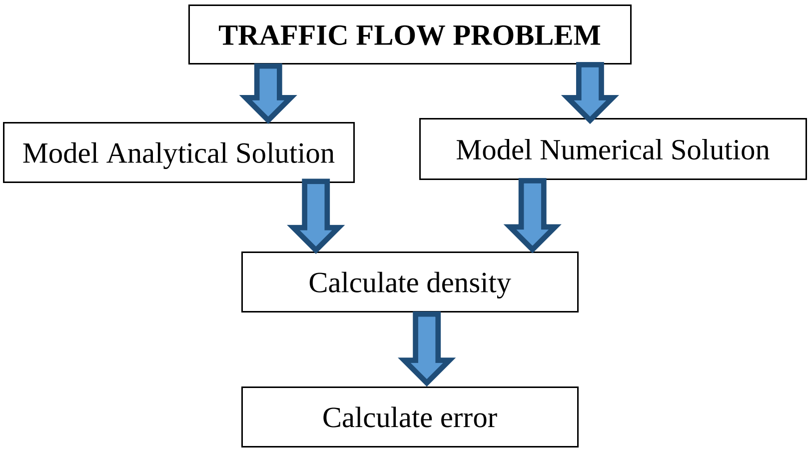

Figure 1: Procedure flow of finding accuracy of finite difference method.

Analytical Solution 0f Traffic Flow Model

Sultana *et al.* (2003) assume all cars have some constant velocity $v > 0$.Then, from the relationship among density, flux and velocity, the flux $q = \rho v$ yields the equation of the continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

in the form

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0. \tag{2}$$

In this case, consider velocity

$$v = v(\rho) \tag{3}$$

as a function of density (Ailkhazraji, 2008). Then, substitute equation (3) into equation (2), equation become

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v(\rho)}{\partial x} = 0. \tag{4}$$

Equation (4) is a first order partial differential equation and nonlinear in ρ. This yield

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{dv}{d\rho}\frac{\partial \rho}{\partial x} = 0.$$

After that,

$$\frac{\partial \rho}{\partial t} + \left(v + \rho\frac{dv}{d\rho}\right)\frac{\partial \rho}{\partial x} = 0 \tag{5}$$

which is linear in derivatives but non-linear in ρ, termed as quasi-linear equation.

$$v(\rho) = v_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^2\right) \tag{6}$$

Kabir*et al.* (2010) use a non-linear speed density relation which is in the form as equation (6). Then, a non-linear speed density relation will give relationship for the traffic flux

$$q(\rho) = \rho v(\rho)$$

$$= v_{max}\left(\rho - \frac{\rho^3}{\rho_{max}^2}\right). \qquad (7)$$

Then, if equation (3.10) is put into the general non-linear model partial differential equation (3.8), the explicit non-linear partial differential equation is obtained as in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho . v_{max}\left(1 - \frac{\rho^2}{\rho_{max}^2}\right)\right) = 0 \qquad (8)$$

with $\rho(t_0, x) = \rho_0(x)$. Initial value problem (8), can be solved by the method of characteristics. The partial differential equation in the initial value problem (8) can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0,$$

where

$$q(\rho) = \rho . v_{max}\left(1 - \frac{\rho^2}{\rho_{max}^2}\right)$$

and its derivative is

$$\frac{dq}{d\rho} = v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right).$$

Therefore,

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

implies

$$\frac{\partial \rho}{\partial t} + v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right)\frac{\partial \rho}{\partial x} = 0.$$

Thus,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} = 0 \qquad (9)$$

after applying the characteristic method on the partial differential equation.Hence,

$$\frac{dx}{dt} = v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right). \qquad (10)$$

Integrating equation (9) yields

$$\frac{d\rho}{dt} = 0,$$

$$\int 1\, d\rho = \int 0\, dt,$$

$$\int d\rho = 0$$

$$\rho(x,t) = \text{constant}\,. \qquad (11)$$

When equation (10) is integrated both sides, we get

$$\int dx = \int v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right) dt$$

and yields

$$x(t) = v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right) t + c. \qquad (12)$$

Next, equation (12) gives

$$x(t) = v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right) t + x^0 \qquad (13)$$

which are the characteristics of the initial value problem (8). From equation (11), it turns to

$$\rho(x,t) = c \qquad (14)$$

since the characteristics through (x, t) also passes through $(x^0, 0)$ and $\rho(x, t) = c$ is constant on this curve, thus initial condition is used to write

$$c = \rho(x,t) \quad = \rho(x^0, 0)$$

$$c = \rho_0(x^0). \tag{15}$$

Equation (14) and (15) yield

$$\rho(x,t) = \rho_0(x^0) \tag{16}$$

while equation (13) gives

$$x^0 = x(t) - v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right)t .$$

Therefore, substitute it into equation (16) gives the form

$$\rho(x,t) = \rho_0\left(x - v_{max}\left(1 - \frac{3\rho^2}{\rho_{max}^2}\right)t\right). \tag{17}$$

Thus, this is the analytical solution of characteristic method for the problem (1).

Numerical Solution of Traffic Flow Model

A finite difference scheme for traffic model problem is:

$$\frac{\partial\rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0, \qquad t_0 \le t \le T, a \le x \le b , \tag{18}$$

with initial condition

$$\rho(x,t_0) = \rho_0(x); \;\; a \le x \le b ,$$

and boundary conditions

$$\rho(a,t) = \rho_a(t); \; t_0 < t \le T$$

$$\rho(b,t) = \rho_b(t); \; t_0 < t \le T$$

where

$$f(\rho) = \rho . v_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^2\right).$$

From Taylor's series, apply forward difference formula in time and write

$$\rho(x,t+k) = \rho(x,t) + k\frac{\partial\rho(x,t)}{\partial t} + \frac{k^2}{2!}\frac{\partial^2\rho(x,t)}{\partial t^2} + \cdots$$

$$\frac{\partial\rho(x,t)}{\partial t} = \frac{\rho(x,t+k) - \rho(x,t)}{k} - 0(k).$$

and thus,

$$\frac{\partial\rho(x_i,t_j)}{\partial t} \approx \frac{\rho_i^{j+1} - \rho_i^j}{\Delta t}, \text{where } k = \Delta t. \tag{20}$$

Then, apply central difference formula in space. From Taylor's series, write

$$f(x+h,t) = f(x,t) + h\frac{\partial f(x,t)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2} - \cdots,$$

$$f(x-h,t) = f(x,t) - h\frac{\partial f(x,t)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2} - \cdots.$$

and thus,

$$\frac{\partial f\left(x_i,t_j\right)}{\partial x} \approx \frac{f_{i+1}^{j} - f_{i-1}^{j}}{2\Delta x}\,. \tag{21}$$

Assume

$$\rho_i^j = \rho\left(x_i,t_j\right)$$

and the uniform grid spacing with step size h and k for space and time respectively:

$$x_{i+1} = x_i + h,$$

and

$$t_{j+1} = t_j + k.$$

Next, substitute (20) and (21) in (18), the discrete version of the non-linear partial difference equation formulates the first order finite difference scheme of the form

$$\frac{\rho_i^{j+1} - \rho_i^j}{\Delta t} + \frac{f_{i+1}^j - f_{i-1}^j}{2\Delta x} = 0$$

$$\rho_i^{j+1} = \rho_i^j - \frac{\Delta t}{2\Delta x}\left(f(\rho_{i+1}^j) - f(\rho_{i-1}^j)\right) \tag{22}$$

where

$$i = 1,2,\dots\dots,N;\;\; j = 0,1,\dots\dots,M-1\,.$$

The difference equation is known as Lax-Friedrichs scheme where

$$f(\rho_i^j) = v_{max}\left(\rho_i^j - \left(\frac{\rho_i^j}{\rho_{max}}\right)^2\right). \tag{23}$$

Thus, this is the numerical solution of the first order finite difference scheme for the problem (18).

Absolute error = $|\,\rho_{\text{exact}} - \rho_{approximate}\,|$

Relative error = $\frac{\text{Absolute error}}{\text{Exact solution}}$

Mean square error = $\frac{1}{n}\sum_{\text{i}=1}^{\text{n}}(\text{Exact} - \text{Approximation})^2$

Results and Discussion

We let $v_{max} = 75$ km per hour, $\rho_{max} = \frac{400\text{cars}}{km}$, $\Delta x = h = 0.1$, $\Delta t = k = 0.001$,

$$0 < x < 1\text{ , and } 0 < t < 0.003$$

with initial condition

$$\rho\,(x,0) = \sin(\pi x)\,,\quad 0 \le x \le 1$$

and boundary condition

$$\rho\,(0,t) = 0\,,\qquad t > 0$$

$$\rho\,(1,t) = 0\,,\qquad t > 0.$$

Table 1: Mean Square Error (MSE).

t	**Mean Square Error**
0.001	0.1448
0.002	0.1455
0.003	0.1463

Table 2: Mean Square Error (MSE).

h	**Mean Square Error**
0.05	0.1326
0.1	0.1325
0.2	0.1322

Based on the experiment that have been carried out, it can be summarized that the different values of step time, k and step space, h gives different value of absolute error, relative error and mean square error. From the results, it give that as the value of k is increased, the mean square error also increase. Therefore, it can be concluded the different values of step time, k give the best approximations when the step time, k is smaller

Besides that, the smaller value of step space, h does not gives good approximation to the analytical values because the mean square error is decreasing due to the increasing value of step space, h. From all finding we obtained, it shown that finite difference method gives better approximation to the analytical solution when the step space, h is smaller.However, the error obtained is big. This solution may not as good as other method to get better accuracy for this problem. As a summary, finite difference method is a good method which is capable to solve the one-dimensional traffic flow equation but for this problem, however it may not adequate enough.

Conclusion

This research is aimed to study on the finite difference method for traffic flow equation. The effect of different step time, k and step space, h has been discussed in this study. In addition, the accuracy of finite difference method is determined by calculating the absolute error, relative error, and mean square error. Later on, the results will be compared with the analytical solution of the traffic flow problem. Moreover, the approximation solution of finite difference method also has been calculated by using MATLAB Distributed Computing R2010a software.

Based on the results obtained, the different value of step time, k and step space, h affect the density of cars. It gives different results of absolute error, relative error and mean square error. As the value of step time, k is decreased, the absolute error, relative error and mean square error are also decreased. Conversely, as the value of step space, h is decreased, the absolute error, relative error and mean square error are increased. Referring to the findings, it can be concluded that the finite difference method is well approximated and gives good results with smaller of step time, k and step space, h. However, the absolute error, relative error and mean square error obtained when compared with analytical solution are big. This problem may suitable to be solved by other method in finite difference.

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