

## AN INTEGRAL EQUATION FOR CONFORMAL MAPPING OF MULTIPLY CONNECTED REGIONS ONTO A CIRCULAR REGION

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**Abstract.** Abstract. An integral equation is presented for the conformal mapping of multiply connected regions of connectivity  $m+1$  onto a circular region. The circular region is bounded by a unit circle, with centre at the origin, and  $m$  number of circles inside the unit circle. The development of theoretical part is based on the boundary integral equation related to a non-homogeneous boundary relationship. An example for verification purpose is given in this paper for the conformal mapping from an annulus onto a doubly connected circular region with centres and radii are assumed to be known.

*Keywords* conformal mapping; boundary integral equation; multiply connected region; circular region

## 1.0 INTRODUCTION

Conformal mapping is a field in which pure and applied mathematics are both involved [19]. The conformal mapping between regions in the complex plane, called as angle-preserving mapping in some books, is of importance in advance complex analysis [21]. The uniqueness of conformal mapping can be understood by the preservation of angle between curves in magnitude as well as in direction. It is worth studying because of its usefulness of giving the idea to transform a given problem region with complicated boundaries onto an equivalent one which has simpler and more manageable boundaries. There are five common types of canonical slit region for which a multiply connected region can be mapped onto [2, 19]. They are the parallel slit region, the circular slit region, the radial slit region, the circular slit disk and the circular slit annuli. Another important canonical region which does not involved slit is the multiply connected circular region.

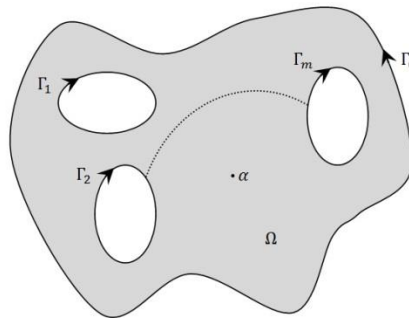
Unfortunately, it is known that the exact conformal map is limited to certain special regions. To deal with this kind of restriction, one has to use numerical approach in constructing such map. Several methods for numerical conformal mapping of multiply connected regions onto canonical slits region have been investigated; approximate conformal mapping methods [4, 8], charge simulation method [20], fast iterative method [10, 18, 24], integral equation method [11, 12, 14, 25, 26] and more [3, 15, 16]. However, the only known numerical method on conformal mapping of multiply connected regions onto multiply circular region is iterative method proposed by Henrici [6] and Luo et al. [9]. Other iterative methods which are proposed by Wegmann [27] and Fornberg-like method [1] are used to compute the inverse mapping. Nasser [17] proposed a fast numerical method which can be implemented on Koebe's classical iterative method efficiently.

This paper gives a construction of an integral equation for a conformal map of multiply connected regions onto a circular region. The construction is based on a boundary integral equation related to a boundary relationship. This approach is inspired by the work of Kerzman and Trummer [7] and Razali et al. [23]. They have derived two integral equations for conformal mapping of simply connected regions onto a disk. Murid and Razali [13] extended the result to doubly connected

case. More work has been followed by Murid and Hu [11] and Sangawi [14, 25] who extended the work to conformal mapping of multiply connected regions onto the five canonical slit regions. Based on the development of this approach, a question now arises whether the same approach can be extended to canonical circular regions. It is the aim of this paper to address the question.

## 2.0 AUXILIARY MATERIALS

### 2.1 Introduction to Problem Region



**Figure 2.1:** An example of a problem region

Let  $\Omega$  be a bounded multiply connected region of connectivity  $m + 1$  in the extended complex plane  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Let the boundary which surrounded  $\Omega$  be denoted as

$$\Gamma = \bigcup_{j=0}^m \Gamma_j, \quad (1)$$

where  $\Gamma_0, \Gamma_1, \dots, \Gamma_m$  are closed smooth Jordan curves. The orientation of each  $\Gamma_j$  is determined such that  $\Omega$  is always on the left as one traces these boundaries. Therefore, the outer curve  $\Gamma_0$  has anti-clockwise orientation. Meanwhile, the orientation for curves  $\Gamma_1, \dots, \Gamma_m$ , which lie in the interior of  $\Gamma_0$ , is in clockwise direction as shown in Figure 2.1. The curves  $\Gamma_j$  are parametrized by  $2\pi$ -periodic twice continuous differentiable complex function denoted as  $z_j = z_j(t)$  for  $t \in J_j = [0, 2\pi]$  with non-zero first derivatives, i.e.,  $z'_j(t) = \frac{dz_j(t)}{dt} \neq 0$ . The total parameter domain  $J$  is the disjoint union of  $m + 1$  interval  $J_0, J_1, \dots, J_m$ .

## 2.2 An Integral Equation Associated to Non-Homogeneous Boundary Relationship

Suppose that  $H(z)$ ,  $c(z)$  and  $Q(z)$  are complex-valued functions defined on  $\Omega \cup \Gamma$  where  $\frac{\overline{H(z)}}{T(z)Q(z)}$  satisfies the Hölder condition and  $H(z) \neq 0$ ,  $c(z) \neq 0$ ,  $Q(z) \neq 0$  for  $z = z(t) \in \Gamma$ . The non-homogeneous boundary relationship is defined as follows [14, 25]:

**Definition 1** Suppose a complex-valued function  $P(z)$  is analytic and single-valued for  $z \in \Omega$  where  $P(z) \neq 0$  for  $z \in \Gamma$ . Suppose further  $P(z)$  is continuous on  $\Omega \cup \Gamma$  and satisfies the non-homogeneous boundary relationship

$$G(z) = \overline{c(z)T(z)Q(z)} \frac{P(z)^2}{|P(z)|^2} + \frac{G(z)H(z)}{\overline{P(z)}}, \quad z \in \Gamma, \quad (2)$$

where a complex-valued function  $G(z)$  is analytic and single-valued for  $z \in \Omega$  with  $G(z) \neq 0$  for  $z \in \Gamma$ , is continuous on  $\Omega \cup \Gamma$  and has a finite number of singularities at  $a_1, a_2, \dots, a_m$  in  $\Omega$ .

The integral equation for analytic function  $P(z)$ , which satisfies the boundary relationship (2) as shown in the following theorem, is derived in [22].

**Theorem 1** Let  $P(z)$  satisfies the boundary relationship (2). Let  $Q(z(t)) = q(t)$  and  $c(z(t)) = \rho(t)$ . Then, for  $z \in \Gamma$

$$\begin{aligned} & P(z) + PV \int_{\Gamma} K(z, w)P(w)|dw| \\ & + c(z)\overline{T(z)Q(z)} \left( \sum_{a_j \text{ inside } \Gamma} \text{Res}_{w=a_j} \frac{P(w)}{(w-z)G(w)} \right)^{\text{conj}} \\ & = -\overline{T(z)Q(z)L(z)}, \quad z \in \Gamma, \end{aligned} \quad (3)$$

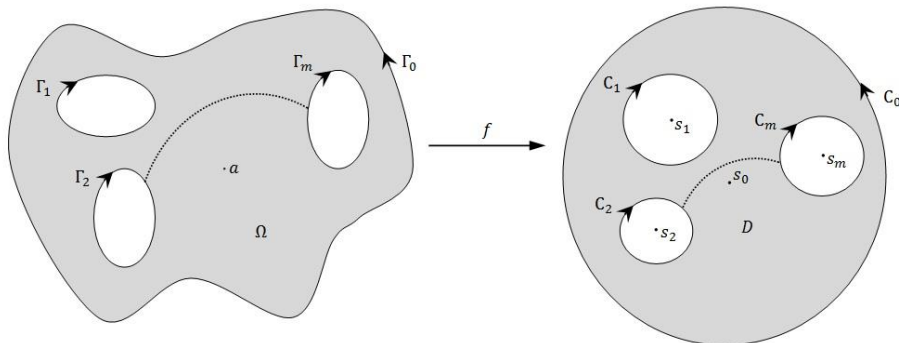
where

$$K(z, w) = \begin{cases} \frac{1}{2\pi i} \left( \frac{c(z)\overline{T(z)Q(z)}}{c(w)\overline{(w-z)Q(w)}} - \frac{T(w)}{w-z} \right), & \text{if } w \neq z, \\ -\frac{1}{2\pi i |z'(t)|} \left( \frac{q'(t)}{q(t)} + \frac{\rho'(t)}{\rho(t)} \right), & \text{if } w = z, \end{cases} \quad (4)$$

$$L(z) = -\frac{1}{2} \frac{H(z)}{Q(z)T(z)} + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{c(z)}H(w)}{c(w)(w-z)Q(w)T(w)} dw. \quad (5)$$

### 3.0 CIRCULAR MAP

For bounded region in Figure 2, a conformal mapping  $f(z)$  from region  $\Omega$  onto a circular region  $D$  exists [6, 27]. Region  $D$  is bounded by  $m + 1$  circles. The centre point of the exterior circle,  $C_0$ , is  $B_0 = 0$  and the radius is  $\mu_0 = 1$ . Meanwhile, the centres  $B_j$  and the radii  $\mu_j$  for inner circles  $C_j$  for  $j = 1, 2, \dots, m$  are unknown and need to be determined. The mapping function  $f$  is called the circular map of  $\Omega$ . The centres and the radii of the circles of region  $D$  are called the parameters of the canonical region  $D$ . We assume  $a$  is a fixed point in  $\Omega$ . The circular region  $D$  as well as conformal mapping  $f$  are uniquely determined by the region  $\Omega$  if  $f$  is normalized by [27]  $f(a) = 0, f'(a) > 0$ .



**Figure 3.1:** An example of a circular map

#### 3.1 The Boundary Values of $f$

Let  $f(z)$  be the analytic function which maps a region  $\Omega$  conformally onto a circular region  $D$ . The boundary values of  $f$  can be represented in the form

$$f(z_j(t)) = B_j + \mu_j e^{i\theta_j(t)}, \quad z_j(t) \in \Gamma_j, \quad t \in [0, 2\pi], \quad (6)$$

for  $j = 0, 1, \dots, m$ , where  $\theta_j(t)$  are the boundary correspondence functions of  $\Gamma_j$ . The boundary values (6) needs to be rearranged into the same form as boundary

relationship (2) mentioned in Section 2, so that the theorem involving integral equation can be applied next. Thus for  $j = 0, 1, \dots, m$ , we can rewrite (6) as

$$f(z_j(t)) = B_j - i\mu_j \text{sign}(\theta'_j(t)) T(z_j(t)) \frac{f'(z_j(t))}{|f'(z_j(t))|} \quad (7)$$

as boundary relationships for conformal mapping which maps a bounded multiply connected region  $\Omega$  into a circular region  $D$ . Note that  $\theta'_0(t) > 0$  as the outer circle  $C_0$  has anti-clockwise orientation. But, for  $j = 1, 2, \dots, m$ ,  $\theta'_j(t) < 0$  as the orientations for inner circles  $C_j$  are in clockwise direction. Here,  $\text{sign}(\theta'_j(t))$  represent  $\frac{|\theta'_j(t)|}{\theta'_j(t)}$  which carries value 1 for  $C_0$  and  $-1$  for  $C_j$ .

### 3.2 Comparison

We rewrite (7) in unified form before we compare it to non-homogenous boundary relationship (2). Let

$$b(z) = \begin{cases} b(z_0) = 0; & z \in \Gamma_0, \\ b(z_j) = B_j; & z \in \Gamma_j, \end{cases} \quad (8)$$

$$\mu(z) = \begin{cases} \mu(z_0) = 1; & z \in \Gamma_0, \\ \mu(z_j) = \mu_j; & z \in \Gamma_j, \end{cases} \quad (9)$$

$$e(z) = \begin{cases} e(z_0) = \text{sign}(\theta'_0(t)) = 1; & z \in \Gamma_0, \\ e(z_j) = \text{sign}(\theta'_j(t)) = -1; & z \in \Gamma_j, \end{cases} \quad (10)$$

for  $j = 1, 2, \dots, m$ . Therefore, the unified form of (7) can be written as

$$f(z) = -i\mu(z)e(z)T(z) \frac{f'(z)}{|f'(z)|} + b(z) \quad (11)$$

with  $z \in \Gamma$ . Now, this form can easily be compared to non-homogeneous boundary relationship (2) which then leads to  $G(z) = f(z)$ ,  $c(z) = i\mu(z)e(z)$ ,  $Q(z) = 1$ ,

$$P(z) = \sqrt{f'(z)}, \text{ and } H(z) = \frac{\overline{\sqrt{f'(z)}}b(z)}{f(z)}.$$

According to Theorem 1, the boundary integral equation related to  $f$  is

$$\sqrt{f'(z)} + \text{PV} \int_{\Gamma} K(z, w) \sqrt{f'(w)} |dw|$$

$$+c(z)\overline{T(z)}\left(\operatorname{Res}_{w=a_j}\frac{\sqrt{f'(w)}}{(w-z)f(w)}\right)^{\operatorname{conj}} = -\overline{T(z)L(z)}, \quad z \in \Gamma, \quad (12)$$

where

$$K(z, w) = \begin{cases} \frac{1}{2\pi i} \left( \frac{c(z)\overline{T(z)}}{c(w)\overline{(w-z)}} - \frac{T(w)}{w-z} \right), & \text{if } w \neq z, \\ 0, & \text{if } w = z, \end{cases} \quad (13)$$

$$L(z) = -\frac{1}{2} \frac{\overline{\sqrt{f'(z)}b(z)}}{f(z)T(z)} + \operatorname{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{c(z)}}{c(w)\overline{(w-z)}} \frac{\overline{\sqrt{f'(w)}b(w)}}{f(w)T(w)} dw, \quad (14)$$

$$c(z) = ie(z)\mu(z), \quad c(w) = ie(w)\mu(w). \quad (15)$$

The residue in (12) can be determined using the fact that  $\operatorname{Res}_{w=w_0}g(w) = \frac{\xi(w_0)}{\zeta'(w_0)}$  [21]. Therefore, since  $\frac{\sqrt{f'(w)}}{(w-z)f(w)}$  has a simple pole at  $w = a$ , thus

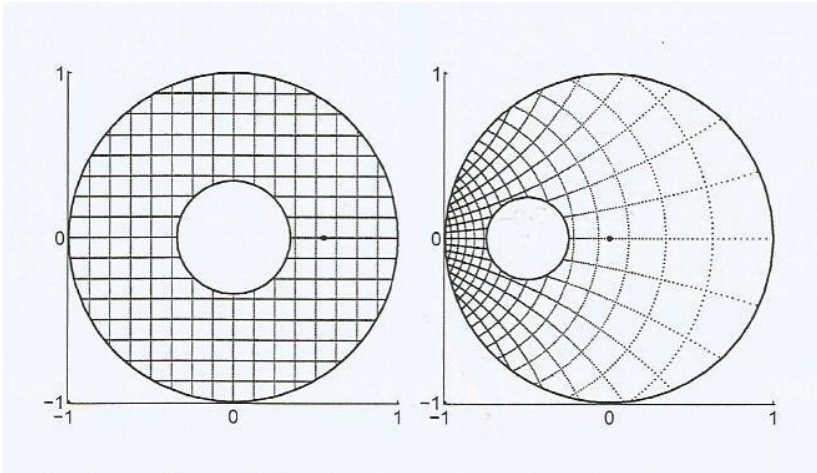
$$\operatorname{Res}_{w=a_j} \frac{\sqrt{f'(w)}}{(w-z)f(w)} = \frac{1}{(a-z)\sqrt{f'(a)}} \quad (16)$$

### 3.3 Exact Mapping Function

We consider one example of known exact mapping function [5]

$$f(z) = \frac{z-a}{1-az} \quad \text{where} \quad a = \frac{16}{19 + \sqrt{105}}, \quad R = \frac{8}{13 + \sqrt{105}} \quad (17)$$

which maps an annulus with boundaries  $\Gamma_0: z_0(t) = e^{it}$  and  $\Gamma_1: z_1(t) = Re^{-it}$  onto a bounded doubly connected circular region with centres  $-0.5, 0$  and radii  $0.25, 1$  respectively, as shown in Figure 3.1. To verify our integral equation, (12) will be separated to two integral equations where  $z = z_0 \in \Gamma_0$  for outer circle and  $z = z_1 \in \Gamma_1$  for inner circle. The information as shown in Table 1 are needed. Then we substitute the exact mapping function (17) into (12) and evaluate on both sides.



**Figure 3.1:** An example of a circular map

**Table 3.1**

$z \in \Gamma_0$		$z \in \Gamma_1$	
$z(t) = e^{it}$	$z'(t) = iz(t)$	$z(t) = Re^{-it}$	$z'(t) = -iz(t)$
$ z(t)  = 1$	$ z'(t)  = 1$	$ z(t)  = R$	$ z'(t)  = R$
$T(z) = iz$	$\overline{T(z)} = -i\bar{z}$	$T(z) = -\frac{iz}{R}$	$\overline{T(z)} = \frac{i\bar{z}}{R}$
$c(z) = 1$	$b(z) = 0$	$c(z) = -0.25i$	$b(z) = 0.5$
$z\bar{z} = 1$	$T(z)\overline{T(z)} = 1$	$z\bar{z} = R^2$	$T(z)\overline{T(z)} = 1$

The integral equation (12) will now be written as

$$\begin{aligned}
 & \sqrt{f'(z)} + c(z)\overline{T(z)} \text{PV} \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\sqrt{f'(w)}\overline{T(w)}}{c(w)(\bar{w} - \bar{z})} dw \\
 & - c(z)\overline{T(z)} \text{PV} \frac{1}{2\pi i} \int_{-\Gamma_1} \frac{\sqrt{f'(w)}\overline{T(w)}}{c(w)(\bar{w} - \bar{z})} dw - \text{PV} \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\sqrt{f'(w)}}{(w - z)} dw \\
 & + \text{PV} \frac{1}{2\pi i} \int_{-\Gamma_1} \frac{\sqrt{f'(w)}}{(w - z)} dw + \frac{c(z)\overline{T(z)}}{(\bar{a} - \bar{z})\sqrt{f'(a)}} = \frac{b(z)\sqrt{f'(z)}}{2\overline{f'(z)}}
 \end{aligned}$$



$$\begin{aligned}
 & -\overline{T(z)}c(z) \left[ \text{PV} \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{c(w)(w-z)T(w)} \frac{b(w)\sqrt{f'(w)}}{f(w)} dw \right]^{\text{conj}} \\
 & +\overline{T(z)}c(z) \left[ \text{PV} \frac{1}{2\pi i} \int_{-\Gamma_1} \frac{1}{c(w)(w-z)T(w)} \frac{b(w)\sqrt{f'(w)}}{f(w)} dw \right]^{\text{conj}}, \quad (18)
 \end{aligned}$$

with the fact that  $\overline{T(w)}dw = |dw|$ . For  $z \in \Gamma_0$ , from (18) we obtain, upon substitution of  $a$  and  $R$ , value 4 on both sides. Meanwhile, for  $z \in \Gamma_1$ , from (18) we obtain

$$R - aRz = -2aR^2 + 4R^2 + (2 - 4a)z,$$

which leads to value  $0.3441 - 0.1883z$  on both sides.

#### 4.0 CONCLUSION

Throughout this study, we have constructed a new integral equation for conformal mapping of multiply connected regions onto a circular region. This new constructed integral equation have been verified by an exact function which conformally maps an annulus onto a doubly connected circular region with assumption that both parameters; centre and radius are known.

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