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# EXACT SOLUTIONS FOR UNSTEADY MHD FREE CONVECTION FLOW WITH TIME DEPENDENT SHEAR STRESS

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## ABSTRACT

In this paper, the effects of MHD on unsteady free convection flow with time dependent shear stress are analyzed. The effects of thermal radiation and porosity on the flow are also studied. Closed-form solutions in general form are obtained by using the Laplace transform technique. The obtained results for velocity and temperature are found to satisfy all the imposed initial and boundary conditions and can be reduced to known solutions from the literature as limiting cases. The velocity profile is presented as a sum of convective and mechanical parts. The effects of shear stress and effective Prandtl number on velocity as well as temperature profiles are presented graphically and discussed in details.

**KEYWORDS:** Free convection; MHD; Shear stress; Laplace transform; Heat transfer.

## INTRODUCTION

The researchers in fluid mechanics usually deal with three types of boundary value problems namely: (a) shear stress on boundary; (b) velocity on boundary and (c) mixed boundary value problems. Amongst these, the problems with shear stress at the boundary are specifically important [1]. It is because, the no slip boundary condition may not be necessarily applicable to flows of some polymeric fluids that can slip or slide on the boundary. Thus, the shear stress boundary condition is particularly meaningful. Bearing in mind the importance of shear stress at the boundary, several researchers have considered it in their problems. However, most of them studied it in the absence of MHD and free convection effects [2]. Recently Rubbab et al. [3] in their pioneering work obtained general solutions for free convection flow of viscous fluid with time dependent shear stress at the boundary. In this continuation, Fetecau et al. [4] established some general solutions for MHD natural convection flow with radiative heat transfer and slip condition over a moving plate.

The MHD flow of an electrically conducting fluid past an impulsively started vertical plate, under the action of a transversely applied magnetic field has been studied in the paper of Soundalgekar and Murty [5]. Radiation effects on MHD flow past an impulsively started infinite isothermal vertical have been presented by Chandakara and

Raj [6]. In the paper of Samiulhaq et al [7], MHD free convection flow of an incompressible viscous fluid past an infinite vertical oscillating plate with uniform heat flux in porous medium has been studied. Quite recently, Khan et al. [8] investigated the effects of time dependent wall shear stress on unsteady MHD conjugate flow in a porous medium with ramped wall temperature. However, no attempt is done so far to study free convection flow of viscous fluid with MHD effects under the condition of time dependent shear stress at the boundary.

**MATHEMATICAL FORMULATION**

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta_T(T-T_\infty)-\frac{\nu}{K}u-\frac{\sigma B_\circ^2}{\rho}u;\quad y,t>0, \tag{1}$$

$$\rho C_p\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial y^2}-\frac{\partial q_r}{\partial y}\quad y,\ t>0, \tag{2}$$

The corresponding initial and boundary conditions are

$$\begin{aligned}&u(y,0)=0, T(y,0)=T_\infty,\qquad \forall\ y>0,\\&\frac{\partial u(0,t)}{\partial y}=\frac{f(t)}{\mu}, T(0,t)=T_w,\quad \forall t>0\\&u(\infty,t)=0, T(\infty,t)=T_\infty,\end{aligned} \tag{3}$$

The radiation heat flux under Rosseland approximation is given by

$$q_r=-\frac{4\sigma^*}{3k_R}\frac{\partial T^4}{\partial y}, \tag{4}$$

It is supposed that the temperature difference within the flow are much small, then Eq. can be linearized by expanding into Taylor series about and neglecting higher order terms takes the form

$$T^4\approx 4T_\infty^3T-3T_\infty^4. \tag{5}$$

Introducing Eq. (5) into Eq. (4) and putting the obtained result in Eq. (2), we get

$$\Pr\frac{\partial T}{\partial t}=\nu\left(1+Nr\right)\frac{\partial^2 T}{\partial y^2};\quad y,t>0, \tag{6}$$

In order to reduce Eqs. (1), (3), and (6) into their non-dimensional forms, we introduce the following dimensionless variables

$$\begin{aligned}&u^*=\frac{u}{U}, T^*=\frac{T-T_\infty}{T_w-T_\infty}, y^*=\frac{U}{\nu}y, t^*=\frac{U^2}{\nu}t,\\&f^*(t^*)=\frac{1}{\rho U^2}f(\frac{\nu}{U^2}t^*),\ K_p=\frac{\nu^2}{U^2}\frac{1}{K},\end{aligned} \tag{7}$$

into Eqs. (1) and (6) and dropping out the " * " notation, it yields

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + GrT - K_p u - Mu, \tag{8}$$

$$\Pr_{eff} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2}, \tag{9}$$

The corresponding dimensionless initial and boundary conditions are

$$\begin{aligned} &u(y,0) = 0, T(y,0) = 0 \quad \forall y \geq 0, \\ &\frac{\partial u}{\partial y}\Big|_{y=0} = f(t), \; T(y,0) = 1, \\ &T(\infty,t) = 0, u(\infty,t) = 0. \end{aligned} \tag{10}$$

**EXACT SOLUTION**

Applying Laplace inverse transform to Eqs. (8) and (9) and using the initial conditions from Eq. (10) , then taking inverse Laplace transform we get

$$u(y,t) = u_c(y,t) + u_m(y,t), \tag{11}$$

where

$$\begin{aligned} u_c(y,t) = {} & \frac{Gr\sqrt{\Pr_{eff}}}{\sqrt{\pi\left(K_p+M\right)\left(\Pr_{eff}-1\right)}} \int_0^t \frac{e^{\frac{K_p+M}{\Pr_{eff}-1}(t-s)-\frac{y^2}{4s}-\left(K_p+M\right)s}}{\sqrt{s}}\, ds \\ & -\frac{Gre^{\frac{K_p+M}{\Pr_{eff}-1}t}}{2\left(K_p+M\right)} e^{y\sqrt{\Pr_{eff}}\sqrt{\frac{K_p+M}{\Pr_{eff}-1}}} \operatorname{erf} c\left(\frac{y\sqrt{\Pr_{eff}}}{2\sqrt{t}} + \sqrt{\frac{K_p+M}{\Pr_{eff}-1}t}\right) \\ & -\frac{Gre^{\frac{K_p+M}{\Pr_{eff}-1}t}}{2\left(K_p+M\right)} e^{-y\sqrt{\Pr_{eff}}\sqrt{\frac{K_p+M}{\Pr_{eff}-1}}} \operatorname{erf} c\left(\frac{y\sqrt{\Pr_{eff}}}{2\sqrt{t}} - \sqrt{\frac{K_p+M}{\Pr_{eff}-1}t}\right) \\ & + \frac{Gr}{K_p+M} \operatorname{erf} c\left(\frac{y\sqrt{\Pr_{eff}}}{2\sqrt{t}}\right) \end{aligned} \tag{12}$$

$$u_m(y,t) = -\frac{1}{\sqrt{\pi}} \int_0^t f(t-s) \frac{e^{-\frac{y^2}{4s}-b_6 s}}{\sqrt{s}}\, ds, \tag{13}$$

correspond to the convective and mechanical parts of velocity,

and

$$T(y,t) = \operatorname{erf} c\left(\frac{y\sqrt{\Pr_{eff}}}{2\sqrt{t}}\right). \tag{14}$$

## LIMITING CASE

In this section we discuss one of the limiting cases of our general solutions
In this case we take the arbitrary function $f(t) = fH(t)$, where $f$ is a dimensionless constant and $H(\cdot)$ denotes the unit step function. After time $t = 0$, the infinite vertical plate applies a constant shear stress to the fluid. The convective part of the velocity remains unchanged while the mechanical part takes the following form

$$u_m(y,t) = -\frac{f}{\sqrt{\pi}} \int_0^t \frac{e^{-\frac{y^2}{4s} - (K_p + M)s}}{\sqrt{s}} ds, \tag{15}$$

equivalently

$$u_m(y,t) = -\frac{f}{\sqrt{K_p + M}} e^{-y\sqrt{b_6}} + \frac{2f}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-\frac{y^2}{4z^2} - (K_p + M)z^2} dz, \tag{16}$$

for $K_p \neq 0, M \neq 0$.

Moreover, if we take $M = 0$, Eq. (15) reduces to the form

$$u_m(y,t) = -\frac{f}{\sqrt{K_p}} e^{-y\sqrt{K_p}} + \frac{2f}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-\frac{y^2}{4z^2} - K_p z^2} dz, \tag{17}$$

which is equivalent to [1, Eq. 28] with the correction of $\sqrt{K_p}$.

Furthermore, in the absence of both $K_p = 0$ and $M = 0$ in Eq. (15), we get

$$u_m(y,t) = -\frac{f}{\sqrt{\pi}} \int_0^t \frac{e^{-\frac{y^2}{4s}}}{\sqrt{s}} ds. \tag{18}$$

which is in good agreement with result reported earliar [2, Eq. 23].

## RESULTS AND DISCUSSION

In order to understand the physical aspects of the problem, the numerical results for velocity and temperature are computed and plotted for various parameters of interest such as magnetic parameter $M$, porosity parameter $K_p$, effective Prandtl number $\Pr_{eff}$, Grashof number $Gr$, dimensionless time $t$ and shear stress $f$. Here we discuss two of them in detail. The effects of the shear stress $f$ induced by the bounding plate on the non-dimensional velocity profiles are shown in Figure. 1. The velocity of fluid is found to decrease with increasing $f$ which is strong agreement with [1]; Figure. 4 .

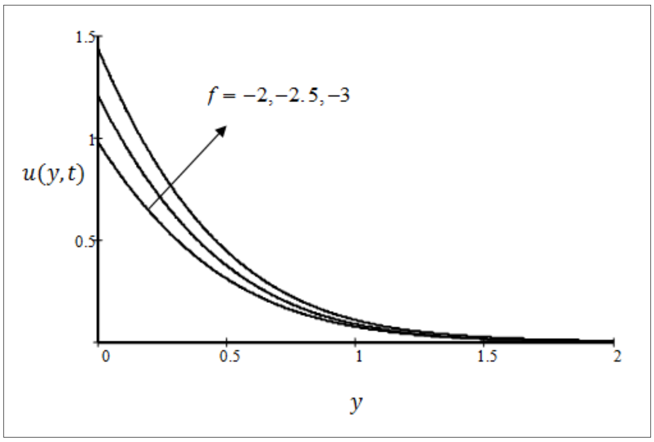


**Figure 1.** Velocity profiles for different values of $f$.

It is found that velocity increases with increasing $t$. Graphical results to show the influence of the effective Prandtl number $\Pr_{eff}$ on velocity profiles are presented in Figure. 2 It is observed that the velocity is a decreasing function with respect to $\Pr_{eff}$. This graphical result agrees with the previously published by [1]; Figure. 2.

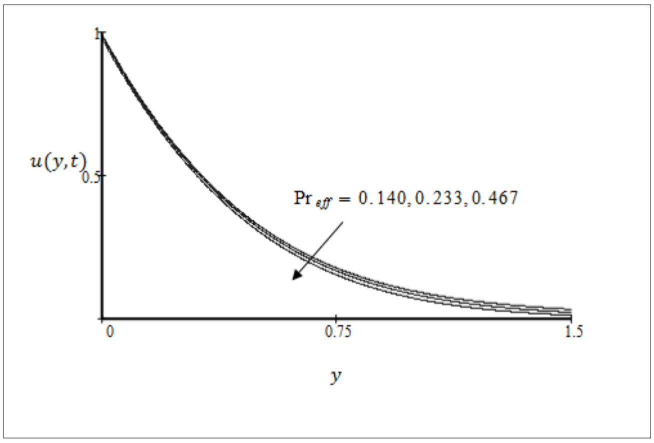


**Figure 2.** Velocity profiles for different values of $\Pr_{eff}$.

The temperature variations against $y$ for various values of effective Prandtl numbers are highlighted in Figure. 3. The significant decrease of the temperature is found as a result of an increase of the effective Prandtl number.

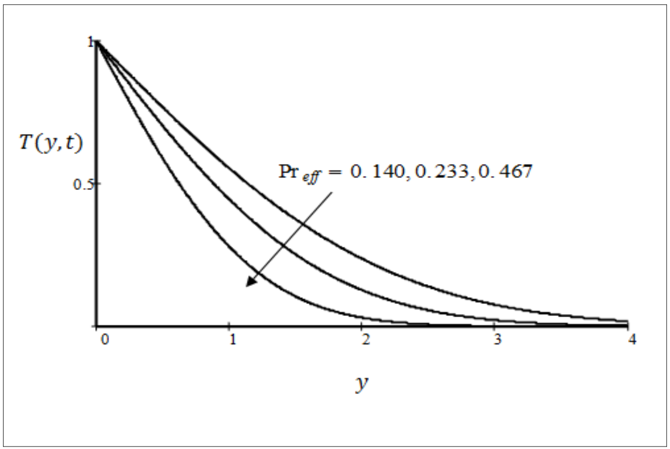


**Figure 3.** Temperature profiles for different values of $\Pr_{eff}$.

## SUMMARY AND CONCLUSION

The purpose of this work is to analyze the unsteady MHD free convection flow of an incompressible viscous fluid over an infinite plate that applies an arbitrary shear stress to the fluid. Exact solutions for velocity, temperature are obtained using the Laplace transform technique and expressed in terms of the complementary error function. It is found that velocity of the fluid $u(y,t)$ can be written as a sum of its convective and mechanical components $u_c(y,t)$, respectively $u_m(y,t)$.

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