

SOLVING DIRICHLET AND NEUMANN PROBLEMS WITH DISCONTINUOUS
COEFFICIENTS ON BOUNDED SIMPLY AND MULTIPLY CONNECTED
REGIONS

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To my family, especially my wife,

"MAHROKH FALLAHGHADI"

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ABSTRACT

Many problems in science and engineering require the solution of the Dirichlet problem and Neumann problem with discontinuous coefficients. In this thesis, a boundary integral equation method is developed for solving Laplace's equation with Dirichlet condition and Neumann condition with discontinuous coefficients in both simply and multiply connected regions. The methods are based on a uniquely solvable boundary linear integral equation with the Neumann kernel. For numerical experiments, discretizing each integral equation leads to a system of linear equations. The system is then solved using the generalized minimum residual method (*gmres*) powered by the fast multipole method (FMM). After the boundary values of the solution of the Dirichlet problem and the Neumann problem with discontinuous coefficients in both simply and multiply connected regions are computed, the solution of the problem at the interior points are calculated by means of the Cauchy integral formula. The numerical examples presented have illustrated that the boundary integral equation methods developed yield high accuracy. Then a method by using these concepts is suggested for solving the mixed boundary value problem. The method is based on converting the mixed problem to a Riemann-Hilbert problem with discontinuous coefficients which is then reduced to two Dirichlet problems, one with discontinuous coefficients and one with unbounded coefficients.

ABSTRAK

Kebanyakan masalah yang terdapat dalam sains dan kejuruteraan memerlukan penyelesaian bagi masalah Dirichlet dan masalah Neumann dengan pekali tidak selanjar. Dalam tesis ini, kaedah persamaan kamiran sempadan digunakan untuk menyelesaikan persamaan Laplace bersama dengan syarat Dirichlet dan syarat Neumann dengan pekali tidak selanjar atas rantau terkait ringkas dan rantau terkait berganda. Kaedah ini berdasarkan persamaan kamiran sempadan linear yang unik penyelesaiannya dengan inti Neumann. Bagi penyelesaian berangka, pendiskretan setiap persamaan kamiran membawa kepada sistem persamaan linear. Sistem ini kemudian diselesaikan dengan menggunakan kaedah reja minimum teritlak (*gmres*) berserta dengan kaedah multikutub pantas (FMM). Setelah nilai sempadan bagi penyelesaian masalah Dirichlet dan masalah Neumann dengan pekali tidak selanjar dalam rantau terkait ringkas dan rantau terkait berganda dikira, penyelesaian pada titik pedalaman pula dikira menggunakan formula kamiran Cauchy. Contoh berangka membuktikan kaedah persamaan kamiran sempadan memberikan penyelesaian dengan ketepatan tinggi. Oleh itu penggunaan kaedah ini dicadangkan dalam mencari penyelesaian kepada masalah nilai sempadan bercampur. Berdasarkan kaedah ini, masalah bercampur ditukarkan kepada masalah Riemann-Hilbert dengan pekali tidak selanjar kemudiannya ditukarkan kepada dua masalah Dirichlet, satu dengan pekali tidak selanjar dan satu dengan pekali yang tidak terbatas.

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LIST OF ABBREVIATIONS

FEM	–	Finite Element Method
BVP	–	Boundary Value Problem
RH	–	Riemann-Hilbert
PDE	–	Partial Differential Equation
BC	–	Boundary Condition
FMM	–	Fast Multipole Method
BEM	–	Boundary Element Method
BIE	–	Boundary Integral Equation
FDM	–	Finite Difference Method
EFM	–	Element-Free (or Meshfree) Method
BNM	–	Boundary Node Method
ODE	–	Ordinary Differential Equation
PC	–	Personal Computer
DOF	–	Degrees of Freedom

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Solving elliptic problems with different types of boundary conditions are unavoidable parts of computational physics and mechanics. There are many problems that arise in engineering and science that require knowledge on Dirichlet boundary value problem. Applications of Dirichlet boundary value problem exist in large number in classical mathematical physics, such as potential of flow around airfoil, heat problem in an insulated plate and electrostatic potential in a cylinder.

Physicists and engineers know about immense applications of such boundary value problems, so formalization of these problems mathematically and obtaining their solutions by using the most appropriate analytical or numerical method are their concern. Since in general the problem cannot be solved in closed form, we have to use the numerical methods to approximate the solution considered. There are many known numerical methods. The common ones are the boundary integral methods, finite elements methods and the finite difference methods that are used by many researchers and also studied extensively.

A classical method for solving boundary value problems is the boundary integral equation method. The classical boundary integral equations for boundary value problems are the second kind Fredholm integral equations with the Neumann kernel. These integral equations are derived by representing the solutions of the boundary value problems as the potential of a single layer [1].

Boundary integral equations will be studied for the Dirichlet problem with discontinuous coefficients at arbitrary simply connected regions as well as for multiply

connected regions in this thesis. These integral equations are Fredholm integral equations of the second kind such that the kernels are continuous.

1.2 Background of the Problem

The boundary integral equation method is a more direct approach that avoid conformal mapping. Recently Wegman et al. [2] and Wegman and Nasser [3] have developed two integral equations with the generalized Neumann kernel that can be used to solve the solution of Riemann-Hilbert boundary value problem. This method is based on earlier works in [2, 4, 5] related to the Riemann-Hilbert problem. The relation between integral equation with the generalized Neumann kernel and Riemann-Hilbert has been studied in these works. The interplay of generalized Neumann kernel and conformal mapping has been investigated in [6, 7, 8, 9]. Generalized Neumann kernel for solving the Riemann-Hilbert problem has been studied in [2, 3, 4, 5, 10]. Solving the mixed Dirichlet-Neumann boundary integral equation with generalized Neumann kernel has been explained in [11, 12, 13]. Dirichlet and Neumann problem with continuous coefficients and generalized Neumann kernel have been solved in [14, 15]. Nevertheless, these approach has not been used for solving the boundary value problems involving first kind of the discontinuous coefficients.

1.3 Problem Statement

The research problem is to reformulate the Dirichlet boundary value problem with discontinuous coefficients on bounded simply connected region as well as on bounded multiply connected region into the form of Dirichlet problem with continuous coefficients and then solve it by using integral equation method with the generalized Neumann kernel. The research problem also includes Neumann problem with discontinuous coefficients and finding a method for solving mixed boundary value problem on simply bounded connected region based on reformulating mixed boundary value problem into the form of Riemann-Hilbert problem with discontinuous coefficients.

1.4 Research Scopes and Objectives

1. To reformulate the Dirichlet condition with discontinuous coefficients on bounded simply and multiply connected regions into the form of Dirichlet problem with continuous coefficients.
2. To determine the integral equations related to the Dirichlet problem with continuous coefficients and then solve it numerically by using the integral equations with the Neumann kernel.
3. To solve Dirichlet boundary value problem with discontinuous coefficients by using the obtained answer of Dirichlet boundary value problem with continuous coefficients and compare it with some existing methods or with exact solutions.
4. To suggest a method for solving the mixed boundary value problem on bounded simply connected region based on reformulating it to a RH problem with discontinuous coefficients.

1.5 Simulation Tool

MATLAB is very powerful tool for doing computations mathematics, specially in the case that our primary data are in the forms of vectors and matrices. It is updated version of software such as Fortran and C/C++ for doing more complicated calculation. For numerical computing, it includes a large number of mathematical functions and toolbox. These powerful numerical capabilities are made in state-of-the-art libraries and LAPACK and BLAS for linear algebra. By mixing these highly advanced functions and toolboxes with additional leading-edge methods, MATLAB affords us to gain a fast, robust, and wide collection of numerical routines available. In addition, MATLAB allows us to customize existing algorithms and make our own algorithms. In this thesis MATLAB R2011a has been used. All calculations were done on Intel(R)Core(TM)i5 CPU M460@2.53GHz 2.53GHz Laptop.

1.6 Thesis Outline

This thesis consists of six chapters including this introductory chapter. Chapter 2 can be regarded as preliminaries with general introduction and formulation, historical

background of the Dirichlet problem and brief review of the some current available methods for solving the Dirichlet problem with discontinuous coefficients.

Chapter 3 explains some results that will be used for solving Dirichlet problem with discontinuous coefficients in bounded simply connected region. First, some preliminaries for solving Dirichlet problem with discontinuous coefficients will be reviewed. Next, the integral equations will be derived and applied to solve this problem. Finally, there will be some numerical examples to prove the accuracy of the suggested method. Neumann problem with discontinuous coefficients will also be discussed in this chapter.

Chapter 4 is motivated by extending the contents of Chapter 3 to solve the Dirichlet problem with discontinuous coefficients in bounded multiply connected region. First, some notations and auxiliary materials will be presented. Next, we solve Dirichlet problem with discontinuous coefficients in bounded multiply connected region. Finally, as in Chapter 3, there are some numerical examples to prove the accuracy of the suggested method.

Chapter 5.3 includes a brief summary of the main results of thesis and some suggestions for future works. Briefly, based on the contents of Chapters 3 and 4, a method will be presented to compute mixed boundary value problem in any arbitrary simply connected region as the following orders: First, some introductions about mixed boundary value problem shall be presented. Next, the mixed boundary value problem will be re-changed to a Riemann-Hilbert problem with discontinuous coefficients. Finally, this Riemann-Hilbert problem with discontinuous coefficients will be solved by decomposing it into two Dirichlet problems, one with discontinuous coefficients and another one that has coefficients with singularity.

Three appendices are attached in this thesis, Appendices A-C. Appendix A presents the list of all papers that have been published and presented during the authors candidature. Appendix B is about some facts related to Sokhotskyi formulas, Hölder condition and Nyström method. In Appendix C, there are some MATLAB programs in related to the some examples that are studied in this thesis.

REFERENCES

1. Henrici, P. *Applied and Computational Complex Analysis*. vol. 3. New York: John Wiley. 1986.
2. Wegmann, R., Murid, A. H. M. and Nasser, M. M. S. The Riemann–Hilbert problem and the generalized Neumann kernel. *J. Comput. Appl. Math.*, 2005. 182(2): 388–415.
3. Wegmann, R. and Nasser, M. M. S. The Riemann–Hilbert problem and the generalized Neumann kernel on multiply connected regions. *J. Comput. Appl. Math.*, 2008. 214(1): 36–57.
4. Nasser, M. M. S. *Boundary Integral Equation Approach for the Riemann Problem*. Ph.D. Thesis. Universiti Teknologi Malaysia. 2005.
5. Murid, A. H. M. and Nasser, M. M. S. Eigenproblem of the generalized Neumann kernel. *Bull. Malaysia. Math. Sci. Soc.(second series)*, 2003. 26: 13–33.
6. Nasser, M. M. S. Numerical conformal mapping of multiply connected regions onto the second, third and fourth categories of Koebe's canonical slit domains. *J. Math. Anal. Appl.*, 2011. 382(1): 47–56.
7. Nasser, M. M. S. Numerical conformal mapping via a boundary integral equation with the generalized Neumann kernel. *SIAM J. Sci. Comput.*, 2009. 31(3): 1695–1715.
8. Nasser, M. M. S. A boundary integral equation for conformal mapping of bounded multiply connected regions. *Comput. Methods Funct. Theory*, 2009. 9(1): 127–143.
9. Nasser, M. M. S. and Al-Shihri, F. A fast boundary integral equation method for conformal mapping of multiply connected regions. *SIAM J. Sci. Comput.*, 2013. 35(3): 1736–1760.
10. Nasser, M. M. S., Murid, A. H. M. and Zamzamir, Z. A boundary integral method for the Riemann–Hilbert problem in domains with corners. *Complex Var. Elliptic Equ.*, 2008. 53(11): 989–1008.

11. Nasser, M. M. S., Murid, A. H. M. and Al-Hatemi, S. A. A. A boundary integral equation with the generalized Neumann kernel for a certain class of mixed boundary value problem. *Journal of Applied Mathematics*, 2012.
12. Al-Hatemi, S. A. A., Murid, A. H. M. and Nasser, M. M. S. A boundary integral equation with the generalized Neumann kernel for a mixed boundary value problem in unbounded multiply connected regions. *Boundary Value Problems*, 2013. 2013(1): 1–17.
13. Al-Hatemi, S. A. A. *Solving Laplace's Equation with Dirichlet-Neumann Condition via Integral Equations with the Generalized Neumann Kernel*. Ph.D. Thesis. Universiti Teknologi Malaysia. 2013.
14. Nasser, M. M. S., Murid, A. H. M., Ismail, M. and Alejaily, E. M. A. Boundary integral equations with the generalized Neumann kernel for Laplace's equation in multiply connected regions. *Applied Mathematics and Computation*, 2011. 217(9): 4710–4727.
15. Nasser, M. M. S. Boundary Integral Equations with the Generalized Neumann Kernel for the Neumann Problem. *MATEMATIKA*, 2007. 23(2): 83–98.
16. Dosiyevev, A. A. and Cival, S. A difference-analytical method for solving Laplace's boundary value problem with singularities. *Dynamical Systems and Applications*, 2004. 33: 339–360.
17. Cooke, J. C. The coaxial circular disc problem. *Z. Angew. Math. Mech.*, 1958. 38: 349–357.
18. Crowdy, D. The Schwarz problem in multiply connected domains and the Schottky–Klein prime function. *Complex Var. Elliptic Equ.*, 2008. 53(3): 221–236.
19. Erdogan, F. *Mixed boundary value problems in mechanics*. vol. 4, chapter 1. Pergamon Press, New York. 1978.
20. Garnett, J. B. and Marshall, D. E. *Harmonic Measure*. Cambridge: Cambridge University Press. 2005.
21. Helsing, J. and Ojala, R. On the evaluation of layer potentials close to their sources. *J. Comput. Phys.*, 2008. 227(5): 2899–2921.
22. Ho, H. S. and Gutierrez-Lemini, D. A general solution procedure in elastostatics with multiply-connected regions with applications to mode III fracture mechanics problems. *Acta Mechanica*, 1982. 44(1-2): 73–89.
23. Kress, R. *Linear Integral Equations*. Springer. 1989.
24. Read, W. W. An analytic series method for Laplacian problems with mixed

- boundary conditions. *J.Comput.Appl.Math*, 2007. 209(1): 22–32.
25. Sykes, J. D. and Brown, R. M. The mixed boundary problem in L_p and Hardy spaces for Laplace's equation on a Lipschitz domain. *Contemporary Mathematics*, 2001. 277: 1–18.
 26. Von Petersdorff, T. and Leis, R. Boundary integral equations for mixed Dirichlet, Neumann and transmission problems. *Mathematical Methods in the Applied Sciences*, 1989. 11(2): 185–213.
 27. Azzam, A. and Kreyszig, E. On solutions of elliptic equations satisfying mixed boundary conditions. *SIAM J. Math. Anal.*, 1982. 13(2): 254–262.
 28. Melnikov, Y. A. and Melnikov, M. Y. Green's functions for mixed boundary value problems in regions of irregular shape. *Electronic Journal of Boundary Elements*, 2007. 4(3).
 29. Gakhov, F. D. *Boundary Value Problems*. Pergamon Press, Oxford. 1966.
 30. Krantz, S. G. *Geometric Function Theory: Explorations in Complex Analysis*. Berlin: Birkhäuser Boston. 2006.
 31. Mikhlin, S. G. *Integral Equations and their Applications to Certain Problems in Mechanics Mathematical Physics and Technology*. Armstrong: Pergamon Press, London. 1957.
 32. Muskhelishvili, N. I. *Singular Integral Equations: Boundary Problems of Function Theory and Their Applications to Mathematical Phys.* Noordhoff. 1977.
 33. Vekua, I. N. and Sneddon, I. N. *Generalized Analytic Functions*. vol. 29. Pergamon Press Oxford. 1962.
 34. Komatu, Y., Mizumoto, H. *et al.* On transference between boundary value problems for a sphere. *Kodai Mathematical Seminar Reports*. Tokyo Institute of Technology, Department of Mathematics. 1954, vol. 6. 115–120.
 35. Martin, P. On the null-field equations for the exterior problems of acoustics. *The Quarterly Journal of Mechanics and Applied Mathematics*, 1980. 33(4): 385–396.
 36. Anh, F. K. Two approximate methods for solving nonlinear Neumann problems. *Ukrainian Mathematical Journal*, 1988. 40(5): 527–532.
 37. Sadiku, M. and Gu, K. A new Monte Carlo method for Neumann problems. 1996: 92–95.
 38. Krutitskii, P. A new integral equation approach to the Neumann problem in

- acoustic scattering. *Mathematical Methods in the Applied Sciences*, 2001. 24(16): 1247–1256.
39. Calìò, F., Marchetti, E., Pavani, R. and Micula, G. About some Volterra problems solved by a particular spline collocation. *Studia Univ. Babeş-Bolyai*, 2003. 48: 135–158.
 40. Kreiss, H. O., Petersson, N. A. and Yström, J. Difference approximations of the Neumann problem for the second order wave equation. *SIAM Journal on Numerical Analysis*, 2004. 42(3): 1292–1323.
 41. Jahanshahi, M. Reduction of the Neumann, Poincare and Robin-Zaremba boundary value problems for Laplace's equation to the Dirichlet boundary value problem. *Appl. Comput. Math*, 2007. 6(1): 51–57.
 42. Liu, Y. *Fast Multipole Boundary Element Method: Theory and Applications in Engineering*. Cambridge university press. 2009.
 43. Atkinson, K. E. *The Numerical Solution of Integral Equations of the Second Kind*. 4. Cambridge university press. 1997.
 44. Kovach, L. D. *Boundary Value Problems*. Addison-Wesley Publishing Company. 1984.
 45. Derrick, W. R. *Complex Analysis and Applications*. Wadsworth International Group. 1984.
 46. Antontsev, S. N., Kazhikto, A. and Monakhov, V. N. *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*. Elsevier. 1989.
 47. Kurt, N., Sezer, M. and Çelik, A. Solution of Dirichlet problem for a rectangular region in terms of elliptic functions. *International Journal of Computer Mathematics*, 2004. 81(11): 1417–1426.
 48. Chen, K., Kao, J., Chen, J., Young, D. and Lu, M. Regularized meshless method for multiply-connected-domain Laplace problems. *Engineering Analysis with Boundary Elements*, 2006. 30(10): 882–896.
 49. Rokhlin, V. Rapid solution of integral equations of classical potential theory. *Journal of Computational Physics*, 1985. 60(2): 187–207.
 50. Greengard, L. and Rokhlin, V. A fast algorithm for particle simulations. *Journal of Computational Physics*, 1987. 73(2): 325–348.
 51. Greengard, L. *The Rapid Evaluation of Potential Fields in Particle Systems*. MIT press. 1988.
 52. Nasser, M. M. S. Fast solution of boundary integral equations with the

- generalized Neumann kernel. *arXiv preprint arXiv:1308.5351*, 2013.
53. Greengard, L. and Gimbutas, Z. FMMLIB2D: A MATLAB toolbox for fast multipole method in two dimensions, Version 1.2, 2012. URL <http://www.cims.nyu.edu/cmcl/fmm2dlib/fmm2dlib.html>.
 54. Kress, R. A Nyström method for boundary integral equations in domains with corners. *Numerische Mathematik*, 1990. 58(1): 145–161.
 55. Peirce, A. Introductory lecture notes on Partial Differential Equations, Lecture 25: More Rectangular Domains: Neumann Problems, mixed BC, and semi-infinite strip problems, 2013. URL http://www.math.ubc.ca/~peirce/M257_316_2012_Lecture_25.pdf.