SOLVING DIRICHLET AND NEUMANN PROBLEMS WITH DISCONTINUOUS COEFFICIENTS ON BOUNDED SIMPLY AND MULTIPLY CONNECTED REGIONS

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To my family, especially my wife,

”MAHROKH FALLAHGHADI”

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ABSTRACT

Many problems in science and engineering require the solution of the Dirichlet problem and Neumann problem with discontinuous coefficients. In this thesis, a boundary integral equation method is developed for solving Laplace's equation with Dirichlet condition and Neumann condition with discontinuous coefficients in both simply and multiply connected regions. The methods are based on a uniquely solvable boundary linear integral equation with the Neumann kernel. For numerical experiments, discretizing each integral equation leads to a system of linear equations. The system is then solved using the generalized minimum residual method (`gmres`) powered by the fast multipole method (FMM). After the boundary values of the solution of the Dirichlet problem and the Neumann problem with discontinuous coefficients in both simply and multiply connected regions are computed, the solution of the problem at the interior points are calculated by means of the Cauchy integral formula. The numerical examples presented have illustrated that the boundary integral equation methods developed yield high accuracy. Then a method by using these concepts is suggested for solving the mixed boundary value problem. The method is based on converting the mixed problem to a Riemann-Hilbert problem with discontinuous coefficients which is then reduced to two Dirichlet problems, one with discontinuous coefficients and one with unbounded coefficients.

ABSTRAK

Kebanyakan masalah yang terdapat dalam sains dan kejuruteraan memerlukan penyelesaian bagi masalah Dirichlet dan masalah Neumann dengan pekali tidak selanjar. Dalam tesis ini, kaedah persamaan kamiran sempadan digunakan untuk menyelesaikan persamaan Laplace bersama dengan syarat Dirichlet dan syarat Neumann dengan pekali tidak selanjar atas rantau terkait ringkas dan rantau terkait berganda. Kaedah ini berdasarkan persamaan kamiran sempadan linear yang unik penyelesaiannya dengan inti Neumann. Bagi penyelesaian berangka, pendiskretan setiap persamaan kamiran membawa kepada sistem persamaan linear. Sistem ini kemudian diselesaikan dengan menggunakan kaedah reja minimum teritlak (`gmres`) berserta dengan kaedah multikutub pantas (FMM). Setelah nilai sempadan bagi penyelesaian masalah Dirichlet dan masalah Neumann dengan pekali tidak selanjar dalam rantau terkait ringkas dan rantau terkait berganda dikira, penyelesaian pada titik pedalaman pula dikira menggunakan formula kamiran Cauchy. Contoh berangka membuktikan kaedah persamaan kamiran sempadan memberikan penyelesaian dengan ketepatan tinggi. Oleh itu penggunaan kaedah ini dicadangkan dalam mencari penyelesaian kepada masalah nilai sempadan bercampur. Berdasarkan kaedah ini, masalah bercampur ditukarkan kepada masalah Riemann-Hilbert dengan pekali tidak selanjar kemudiannya ditukarkan kepada dua masalah Dirichlet, satu dengan pekali tidak selanjar dan satu dengan pekali yang tidak terbatas.

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LIST OF ABBREVIATIONS

FEM	–	Finite Element Method
BVP	–	Boundary Value Problem
RH	–	Riemann-Hilbert
PDE	–	Partial Differential Equation
BC	–	Boundary Condition
FMM	–	Fast Multipole Method
BEM	–	Boundary Element Method
BIE	–	Boundary Integral Equation
FDM	–	Finite Difference Method
EFM	–	Element-Free (or Meshfree) Method
BNM	–	Boundary Node Method
ODE	–	Ordinary Differential Equation
PC	–	Personal Computer
DOF	–	Degrees of Freedom

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Solving elliptic problems with different types of boundary conditions are unavoidable parts of computational physics and mechanics. There are many problems that arise in engineering and science that require knowledge on Dirichlet boundary value problem. Applications of Dirichlet boundary value problem exist in large number in classical mathematical physics, such as potential of flow around airfoil, heat problem in an insulated plate and electrostatic potential in a cylinder.

Physicists and engineers know about immense applications of such boundary value problems, so formalization of these problems mathematically and obtaining their solutions by using the most appropriate analytical or numerical method are their concern. Since in general the problem cannot be solved in closed form, we have to use the numerical methods to approximate the solution considered. There are many known numerical methods. The common ones are the boundary integral methods, finite elements methods and the finite difference methods that are used by many researchers and also studied extensively.

A classical method for solving boundary value problems is the boundary integral equation method. The classical boundary integral equations for boundary value problems are the second kind Fredholm integral equations with the Neumann kernel. These integral equations are derived by representing the solutions of the boundary value problems as the potential of a single layer [1].

Boundary integral equations will be studied for the Dirichlet problem with discontinuous coefficients at arbitrary simply connected regions as well as for multiply connected regions in this thesis. These integral equations are Fredholm integral equations of the second kind such that the kernels are continuous.

1.2 Background of the Problem

The boundary integral equation method is a more direct approach that avoid conformal mapping. Recently Wegman et al. [2] and Wegman and Nasser [3] have developed two integral equations with the generalized Neumann kernel that can be used to solve the solution of Riemann-Hilbert boundary value problem. This method is based on earlier works in [2, 4, 5] related to the Riemann-Hilbert problem. The relation between integral equation with the generalized Neumann kernel and Riemann-Hilbert has been studied in these works. The interplay of generalized Neumann kernel and conformal mapping has been investigated in [6, 7, 8, 9]. Generalized Neumann kernel for solving the Riemann-Hilbert problem has been studied in [2, 3, 4, 5, 10]. Solving the mixed Dirichlet-Neumann boundary integral equation with generalized Neumann kernel has been explained in [11, 12, 13]. Dirichlet and Neumann problem with continuous coefficients and generalized Neumann kernel have been solved in [14, 15]. Nevertheless, these approach has not been used for solving the boundary value problems involving first kind of the discontinuous coefficients.

1.3 Problem Statement

The research problem is to reformulate the Dirichlet boundary value problem with discontinuous coefficients on bounded simply connected region as well as on bounded multiply connected region into the form of Dirichlet problem with continuous coefficients and then solve it by using integral equation method with the generalized Neumann kernel. The research problem also includes Neumann problem with discontinuous coefficients and finding a method for solving mixed boundary value problem on simply bounded connected region based on reformulating mixed boundary value problem into the form of Riemann-Hilbert problem with discontinuous coefficients.

1.4 Research Scopes and Objectives

1. To reformulate the Dirichlet condition with discontinuous coefficients on bounded simply and multiply connected regions into the form of Dirichlet problem with continuous coefficients.
2. To determine the integral equations related to the Dirichlet problem with continuous coefficients and then solve it numerically by using the integral equations with the Neumann kernel.
3. To solve Dirichlet boundary value problem with discontinuous coefficients by using the obtained answer of Dirichlet boundary value problem with continuous coefficients and compare it with some existing methods or with exact solutions.
4. To suggest a method for solving the mixed boundary value problem on bounded simply connected region based on reformulating it to a RH problem with discontinuous coefficients.

1.5 Simulation Tool

MATLAB is very powerful tool for doing computations mathematics, specially in the case that our primary data are in the forms of vectors and matrices. It is updated version of software such as Fortran and C/C++ for doing more complicated calculation. For numerical computing, it includes a large number of mathematical functions and toolbox. These powerful numerical capabilities are made in state-of-the-art libraries and LAPACK and BLAS for linear algebra. By mixing these highly advanced functions and toolboxes with additional leading-edge methods, MATLAB affords us to gain a fast, robust, and wide collection of numerical routines available. In addition, MATLAB allows us to customize existing algorithms and make our own algorithms. In this thesis MATLAB R2011a has been used. All calculations were done on Intel(R)Core(TM)i5 CPU M460@2.53GHz 2.53GHz Laptop.

1.6 Thesis Outline

This thesis consists of six chapters including this introductory chapter. Chapter 2 can be regarded as preliminaries with general introduction and formulation, historical background of the Dirichlet problem and brief review of the some current available methods for solving the Dirichlet problem with discontinuous coefficients.

Chapter 3 explains some results that will be used for solving Dirichlet problem with discontinuous coefficients in bounded simply connected region. First, some preliminaries for solving Dirichlet problem with discontinuous coefficients will be reviewed. Next, the integral equations will be derived and applied to solve this problem. Finally, there will be some numerical examples to prove the accuracy of the suggested method. Neumann problem with discontinuous coefficients will also be discussed in this chapter.

Chapter 4 is motivated by extending the contents of Chapter 3 to solve the Dirichlet problem with discontinuous coefficients in bounded multiply connected region. First, some notations and auxiliary materials will be presented. Next, we solve Dirichlet problem with discontinuous coefficients in bounded multiply connected region. Finally, as in Chapter 3, there are some numerical examples to prove the accuracy of the suggested method.

Chapter 5.3 includes a brief summary of the main results of thesis and some suggestions for future works. Briefly, based on the contents of Chapters 3 and 4, a method will be presented to compute mixed boundary value problem in any arbitrary simply connected region as the following orders: First, some introductions about mixed boundary value problem shall be presented. Next, the mixed boundary value problem will be re-changed to a Riemann-Hilbert problem with discontinuous coefficients. Finally, this Riemann-Hilbert problem with discontinuous coefficients will be solved by decomposing it into two Dirichlet problems, one with discontinuous coefficients and another one that has coefficients with singularity.

Three appendices are attached in this thesis, Appendices A-C. Appendix A presents the list of all papers that have been published and presented during the authors candidature. Appendix B is about some facts related to Sokhotskyi formulas, Hölder condition and Nyström method. In Appendix C, there are some MATLAB programs in related to the some examples that are studied in this thesis.

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