DETERMINATION OF THE INTERSECTION OF PROJECTIVE PLANE $p^2(k)$ CURVES BASED ON THE BEZOUT'S THEOREM

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Specially dedicated to my lovely and supporting family

To my beloved Papa and Mama,

My lovely brothers, Dennis Torio & Tommy Lukas

Who support me and help me in everything

&

All my friends,

Thanks for all the motivation and hope given so far

To all the UTM lecturers

Thanks for all the knowledge.

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ABSTRACT

The question on the number of points of intersection of curves had been the subject of conjecture for many years. One of the theorems that arise from such conjectures is the Bezout's theorem. Bezout gives a rigorous proof that when two polynomials in two variables are set equal to 0 simultaneously, one of degree m and the other of degree n, then there cannot be more than mn solutions unless the two polynomials have a common factor. This is a first form of Bezout's theorem. In order to have a chance of obtaining a full complement of mn solutions, there have been several adjustments to the first form of Bezout's theorem, which allow complex solutions instead of just real solutions and considering projective plane curves instead of ordinary plane curves to allow for solutions at infinity. This condition can be realized in the example of two lines, which have no points of intersection on the affine plane if they are parallel. However, in the projective planes, parallel lines do intersect, at a point of the line at infinity. This motivates generalization of the Bezout's theorem in the number of intersection of projective plane curves under certain conditions. In the case of two variables conics in the affine plane, the resultant can be applied to solve the intersection points. The notion and properties of intersection multiplicity is then applied on each of the intersection points including the points at infinity by considering the homogenization variable. By letting the homogenization variable equals to 1, the properties of intersection multiplicity can be applied to determine the multiplicity of the affine intersection points. By letting the homogenization variable equals to 0, the intersection multiplicity of the points at infinity can also be determined, thus the implementation of the Bezout's theorem has been illustrated using selected examples.

ABSTRAK

Persoalan kepada bilangan titik persilangan telah menjadi subjek konjektur selama bertahun-tahun. Salah satu teorem yang didapati boleh digunakan untuk mencari bilangan titik persilangan tersebut ialah teorem Bezout. Bezout memberikan bukti yang kukuh dengan menyatakan bahawa apabila dua polinomial dalam dua pembolehubah masing-masing berdarjah m dan n maka tidak boleh wujud lebih daripada mn penyelesaian titik persilangan kecuali kedua-dua polinomial mempunyai faktor yang sama. Ini adalah bentuk pertama teorem Bezout. Dalam usaha untuk mendapatkan penyelesaian yang lengkap, terdapat beberapa pelarasan yang dapat memberikan penyelesaian nombor kompleks dan bukan hanya penyelesaian nombor nyata, serta mempertimbangkan lengkung satah unjuran yang merupakan lanjutan daripada lengkung satah biasa bagi menentukan penyelesaian pada infiniti. Perkara ini dapat dijelaskan bagi contoh dua garisan, yang tidak mempunyai titik persilangan pada satah afin jika kedua-duanya adalah selari. Walau bagaimanapun, dalam satah unjuran, garis selari juga mempunyai titik persilangan di titik garis pada infiniti. Keterangan tentang sifat ini menjadi motivasi pengaplikasian teorem Bezout dalam menentukan bilangan persilangan lengkung pada satah unjuran tertakluk kepada syarat-syarat tertentu. Dalam kes persamaan lengkung dua pembolehubah berdarjah dua yang ditakrifkan dalam satah afin, kaedah resultan boleh digunakan untuk menyelesaikan titik-titik persilangan. Definisi dan sifat-sifat kegandaan persilangan kemudian diaplikasikan pada setiap titik persilangan termasuk titik-titik di infiniti dengan mempertimbangkan pembolehubah homogen. Dengan menetapkan pembolehubah homogen sama dengan 1, sifat-sifat kegandaan persilangan boleh digunakan untuk menentukan kepelbagaian titik-titik persilangan afin. Dengan menetapkan pembolehubah penyeragaman sama dengan 0, kegandaan persilangan bagi titik pada infiniti juga boleh ditentukan. Seterusnya itu pelaksanaan teorem Bezout telah dterangkan dengan contoh-contoh yang dipilih.

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LIST OF SYMBOLS/ABBREVATIONS

SYMBOLS

$A^1(k)$	Affine line
A^2	Affine plane
BKK bound	The root count developed by Bernshtein, Kushnirenko and khovanskii
C	A subset of $p^2(k)$, known as $V(F)$
#	Means the cardinality of the set, example $\# f \cap g = 1$
Cod	Codimension is a term used in a number of algebraic to indicate the difference between the dimension of certain objects and the dimension of a smaller object contained in it, applied in vector space.
$:=$	It is symbol of definition, like $i_0 := \min\{i : g_i \neq 0\}$, means i_0 is defines to be another name for $\min\{i : g_i \neq 0\}$, under certain assumptions taken in context
$\deg(F) \cdot \deg(G)$	The number of intersections of two plane projective curves
f	Nonconstant polynomial
F, G	Nonconstant form of homogeneous projective plane curve

$\mapsto$	It is the function arrow where it maps from the one element to the another element
H_p	The Hessian polynomial of C
I_p	Intersection Multiplicity
ϑ_p	Ring of rational function.
k	A field
L_1, L_2	Projective lines in $p^2(R)$
$p^2(k)$	Projective plane curve
$p^n(k)$	Projective n space over k
$\leftrightarrow$	Iff or proportional logic of equivalence the two variables between are equal.
$R_{F,G}$	The resultant of F and G
R_P	The local ring of rational functions of zero degree at the point P
$\sim$	It shows the similarity between two variables
$V(f)$	The set of zeros of, which is the hypersurface defined by f
V	A finite dimensional vector space over k
x, y	Coordinates of point in algebraic plane curve
$Z(f)$	The curve of f is the subset of P^2 defined as the zero locus

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

Mathematicians studying some topics spent their time figuring out how to construct new equations from old. One of the mathematicians, Descartes' 1637 in La Geometrie [1] focused on the theory and construction of equations. For instance, the paradigm construction of an equation was given by Menaechmus'[1] via the intersection of a hyperbola and a parabola. In other words, mathematicians have used two degree two curves to solve a degree three curve (i.e. $x^3 = 2$).

According to [1], Descartes and Fermat independently found a more general way of solving any degree three or four equations by finding the intersection of a parabola and a circle. Unlike Fermat, Descartes published his results in La Geometrie and further published a particular cubic curve, the Cartesian parabola, whose intersection with a suitable circle gave the solution of any given fifth or sixth degree equation.

In search of a general construction for finding solutions to *nth* degree equations, the mathematicians of the time assumed that the intersection of a degree n curve and a degree m curve consists of nm points. As mentioned by Fulton [1], "it seems that the first mathematician to formally state this assumption was Newton in his 1665 work,

Principia Mathematica: For y^e number of points in wch two lines may intersect can never be greater $y^n y^e$ rectangle of y^e numbers of their dimensions, and they always intersect in so many points, excepting those wch are imaginary only''. This statement is eventually known as Bezout's theorem which states that the number of points of intersection of two distinct irreducible algebraic curves equals the product of their degrees.

1.2 Motivation

A first form of Bezout's theorem is an application which states that if $f(x, y)$ and $g(x, y)$ are polynomials of respective degrees m and n whose locus of common zeros has more than mn points, then f and g have a nontrivial common factor. This version of the theorem regarded as pertaining to a pair of affine plane curves.

In affine plane curves, Bezout's theorem holds only after certain amendments. The first of these is the requirement that we consider points with coordinates in an algebraically closed field. Even if we consider points with coordinates in an algebraically closed field and take account of multiplicities of intersections, this fails in very simple cases, and still needs one further amendment. This can already be seen in the example of two lines, which have no points of intersection if they are parallel. However, on the projective plane, parallel lines do intersect, in a point of the line at infinity.

The projective plane curves, which are non-constant homogenous polynomials in three variables, two such being regarded as the same if they are multiples of one another. If F and G are two projective plane curves of respective degrees m and n over an algebraically closed field, then either they have a nontrivial common factor or they have exactly mn common zeros when the intersection multiplicities of the zeros are

taken into account. Versions of the resultant and Bezout's Theorem are valid in this context, and two projective plane curves defines over an algebraically closed field always have common zeros.

The question on the number of points of intersection had been the subject of conjecture for some time earlier, and it was expected that two plane curves of respective total degrees *m* and *n* in some sense had mn points of intersection [1]. Etienne Bezout (1730-1783) took up this question and dealt with parts of it rigorously. The quadratic case can be solved by finding one variable in terms of the other by substitution. If each polynomial is quadratic in *y* and having coefficients that depend on *x*, then we have a system

$$a_0 + a_1 y + a_2 y^2 = 0$$

$$b_0 + b_1 y + b_2 y^2 = 0 .$$

Instead of regarding this as a system of two equations for the variables $x_{0,} x_{1,} x_{2,}$, where $x_0 = 1, x_1 = y, x_2 = y^2$, we can get two further equations by multiplying each equation by *y*:

$$a_0 + a_1 y + a_2 y^2 + a_3 y^3 = 0$$

$$b_0 + b_1 y + b_2 y^2 + b_3 y^3 = 0$$

This gives 4 homogeneous linear equations for $x_0 = 1, x_1 = y, x_2 = y^2, x_3 = y^3$. Since the system has the nonzero solution $(1, y, y^2, y^3)$, the determinant of the coefficient matrix must be 0. Remembering that the coefficients depend on *x*, we see that we have eliminated the variable *y* and obtained a polynomial equation for *x* without using any solution formula for polynomials in one variable. The device that Bezout introduced for this purpose is the determinant of the coefficient matrix which is called the resultant of the system and is a fundamental tool in handling simultaneous polynomial equations.

Later in 1779, Bezout gives a rigorous proof that when two polynomials in two variables are set equal to 0 simultaneously, one of degree *m* and the other of degree *n*, then there cannot be more than mn solutions unless the two polynomials have a common factor. This is a first form of Bezout's theorem. In order to have a chance of obtaining a full complement of mn solutions, he made three adjustments which allowed complex solutions instead of just real solutions and consider projective plane curves instead of ordinary plane curves to allow for solutions at infinity. This can already be seen in the example of two lines, which have no points of intersection on the affine plane if they are parallel. However, on the projective plane, parallel lines do intersect, in a point of the line at infinity [2].

Since the application of Bezout's theorem on the intersection of curves in the affine space has its limitations when only affine solutions are considered, its application on the intersection of projective plane curve is investigated. Therefore, in this dissertation the following research questions are addressed.

1.3 Research Problem

1) What is an affine space and a projective space?
2) What is the definition of an algebraic plane curve and can we determine the affine and projective intersection points of these curves?
3) How can Bezout's Theorem be applied or implemented on the intersection of projective plane curve $p^2(k)$?

1.4 Objectives of the Study

The following objectives are designed to answer the research problem:

1) To explain the notion of affine and projective space, algebraic plane curves, affine and projective intersection points of the curves.
2) To describe related concepts of Bezout's theorem such as the resultant of polynomial equations and intersection multiplicity.
3) To illustrate the application or implementation of Bezout's theorem on the intersection of projective curves.
4) To provide examples that explain the definition, theorems or difficult concepts involved in achieving the understanding of Bezout's Theorem.

In objective 4), some of the examples are cited. As addition, the detailed working explained in these example.

1.5 Scope of the Study

The study investigates the theory of Bezout when applied to the intersection of the algebraic curves such as projective curve $p^2(k)$. It covers related concepts of Bezout's such as resultant of polynomials and intersection multiplicity. Some examples of the applications of Bezout's theorem are also given, to enhance the understanding of related concepts and theorems.

1.6 Significance of the Study

The algebraic equations are the basis of a number of areas of modern mathematics. Writing and solving equations is an important part of mathematics. Algebraic equations can help us to model situations and solve problems in which quantities are unknown. The simplest type of algebraic equation is a linear equation that has just one variable. Algebraic equations contain variables, symbols that stand for an unknown quantity.

Variables are often represented with letters, like x, y, or z. Sometimes a variable is multiplied by a number. This number is called the coefficient of the variable. An important property of equations is one that states that you can add the same quantity to both sides of an equation and still maintain an equivalent equation.

Geometrically the solutions of this research are the intersection points of the algebraic curves represented by the equations. Thus the research area is algebraic geometry which plays an important role as the theoretical basis to researches related in solving multivariate polynomial equations involving equations with more than one variable.

This study illustrates how Bezout's theorem can be applied to the problem of solving algebraic equations. The theorem has initiated further developments in this field of research which seeks for better bounds on the number of finite solutions to multivariate polynomial equations such as the Bernstein Theorem and BKK bounds.

1.7 Organization of the Study

The organization of the study is divided and arranged according to the respective chapters. The first chapter discusses the background of the research, motivation and research problem. The research problem is addressed by setting the research objectives. The significance of the study and its scope are also included in the first chapter followed by the organization of the dissertation.

Chapter 2 defines the notion of algebraic plane curve, affine space, projective space and projective plane curves. The curves and divisor of homogeneous polynomial are also defined, since Bezout's theorem is about curves in the projective plane we need these definitions and properties in order to formulate and prove the theorem.

After detailed explanation of algebraic plane curves and projective planes curves, then we can proceed to chapter 3 which gives a detailed definition and properties of resultant of polynomials and intersection multiplicity of these curves, and illustrate some examples for a better understanding of related and useful concepts.

Chapter 4 discusses the Bezout’s theorem, the definition and theorem related are defined. In this chapter, the resultant of polynomial and intersection multiplicity are applied to the projective curves in order to fulfill the Bezout’s theorem. Furthermore, the applications of the theorem on the intersection of projective plane curves especially on the intersection of conics and nonsingular cubics in the projective plane of dimension two are explained in this chapter. Some examples are also provided to better comprehend the implementation of the theorem and related concepts.

Lastly, chapter 5 presents the summary and conclusions of the study. The summary of methodology of this study is listed in figure below:

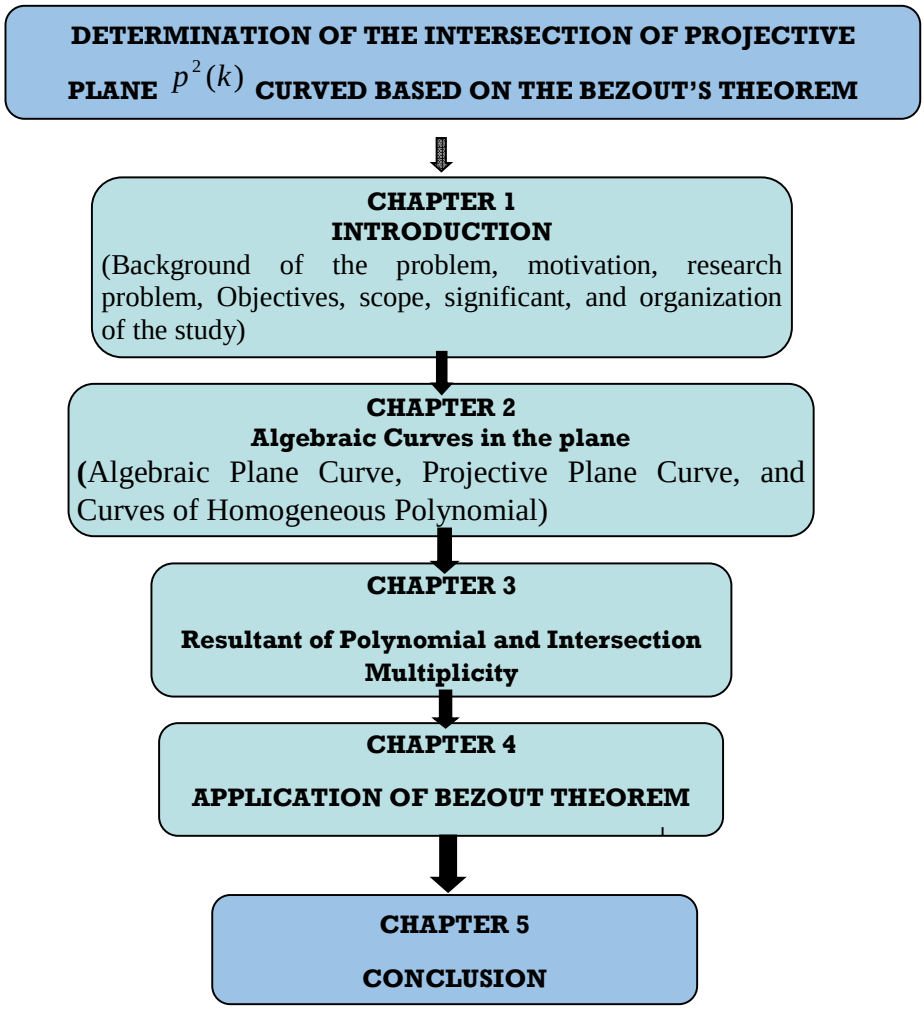

Figure 1.1. Methodology of dissertation

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