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A Metric Discrepancy Estimate for A Real Sequence

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Abstract A general metrical result of discrepancy estimate related to uniform distribution is proved in this paper. It has been proven by J.W.S Cassel and P.Erdos & Koksma in [2] under a general hypothesis of $(g_n(x))_{n=1}^{\infty}$ that for every $\varepsilon > 0$,

$$D(N,x) = O(N^{\frac{-1}{2}}(\log N)^{\frac{5}{2}+\varepsilon})$$

for almost all x with respect to Lebesgue measure. This discrepancy estimate was improved by R.C. Baker [5] who showed that the exponent $\frac{5}{2}+\varepsilon$ can be reduced to $\frac{3}{2}+\varepsilon$ in a special case where $g_n(x) = a_n x$ for a sequence of integers $(a_n)_{n=1}^{\infty}$. This paper extends this result to the case where the sequence $(a_n)_{n=1}^{\infty}$ can be assumed to be real. The lighter version of this theorem is also shown in this paper.

Keywords Discrepancy, uniform distribution, Lebesgue measure, almost everywhere

1 Introduction

The idea of this paper originated from the notion of the uniformly distributed sequences. To differentiate between a good and a bad uniform distribution of a sequence, we need to have a quantitative measure, a discrepancy. A discrepancy of a sequence measures how much a given sequence deviate from an ideal sequence, where it gives us a picture on how good or bad the sequence is distributed. In this paper, we present a result on discrepancy estimate for a real sequence, which is described below. The culmination of this paper is shown in theorem 1. This paper is organized as follows. First we will show the proof of Theorem 1 in section 2 and proof of Theorem 2 in section 3.

Let $(\lambda_n)_{n\geq 0}$ be a sequence of real numbers and there exist $\delta > 0$ such that $|\lambda_{n+1} - \lambda_n| \geq \delta$. Also let

$$D(N,x) = D(\{\lambda_1 x\}, ..., \{\lambda_N x\}) \qquad (n = 1, 2, ...)$$

be the discrepancy of the sequence $(\lambda_n x)_{n=1}^{\infty}$, and $\{\lambda_n x\}$ denotes the fractional part of $\lambda_n x$. It has been proven in [3] that for almost real numbers x with respect to Lebesgue measure $(\lambda_n x)_{n=1}^{\infty}$ is uniformly distributed.

Theorem 1 *Given* $\varepsilon > 0$,

$$D(N,x) = o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}+\varepsilon}) \qquad \text{a.e (almost everywhere).}$$

In the case for dimension $d = 1$, Theorem 1 is due to R.C.Baker [5]. His proof hinges on maximal inequality for partial sums of Fourier series, due to L. Carleson and Hunt, used in the proof of their celebrated theorem that if $c > 1$ and $f \in L^c([0,1))$, then the partial sums of the Fourier series of f converge almost everywhere in $[0,1]$, with respect to Lebesgue measure. This maximal inequality has a generalization to higher dimensions due to P.Sjolin [4], which allows R.Nair [7] to extend Baker's theorem to the case where $d > 1$, but still for only limited type of sequence which only consume the integers. Theorem 2, in the case for $d = 1$, appears in [6] but with the exponent $3\frac{1}{2} + \varepsilon$ instead of $\frac{5}{2} + \varepsilon$ as what we have.

2 Proof of Theorem 1

To prove the theorem, we need the following lemmas.

Lemma 1 [4] : *For a set of real numbers $x_1, ..., x_N$, there exists $C > 0$ such that for all natural numbers L,*

$$ND(x_1, ..., x_N) \leq C \left(\frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Lemma 2 [5] : *Suppose we are given $\delta > 0$, real numbers $(\lambda_n)_{n=1}^N$ such that $\lambda_{n+1} - \lambda_n \geq \delta > 0$, real numbers T and T_0 with $T > 0$ and complex numbers $(a_n)_{n=1}^N$. Then there exists $C > 0$ such that*

$$\int_{T_0}^{T_0+T} \left\{ \max_{1 \leq v \leq N} \left| \sum_{n=1}^{v} a_n e^{i\lambda_n t} \right|^2 \right\} dt \quad \leq \quad C(T + 2\pi\delta^{-1}) \sum_{n=1}^{N} |a_n|^2$$

Lemma 3 (Borel-Cantelli) : *Let μ be a measure on a set X, with σ - algebra F, and let (A_n) be a sequence in F. If*

$$\sum_{n=1}^{\infty} \mu\left(A_n\right) \;<\; \infty,$$

then $\mu(\limsup A_n) = 0$.

Plainly in proving Theorem 1, we may without loss of generality assume x belongs to some finite interval. Now let

$$||f|| \text{ denotes } \left(\int_0^1 |f|^2 \, dx \right)^{\frac{1}{2}}.$$

Then putting $x_n = \lambda_n x$, using Lemma 1 in light of Minskowski's inequality, there exists $C > 0$ such that

$$\left\| \max_{1 \leq v \leq N} vD(v, x) \right\| \;\leq\; C \;\left(\frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} \left\| \max_{1 \leq v \leq N} \left| \sum_{n=1}^{v} e^{2\pi i h \lambda_n x} \right| \right\| \right),$$

which in light of Lemma 2 is

$$\leq \; C_2 \left(\frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} N^{\frac{1}{2}} \right),$$

which choosing $L = N$ is

$$\left\| \max_{1 \leq v \leq N} \; D(v, x) \right\| \leq \; C_2 N^{\frac{1}{2}} (\log N). \tag{*}$$

To deduce Theorem 1 let

$$E(\varepsilon) = \left\{ x \in [T_0, T_0 + T] : \limsup_{l \to \infty} \frac{lD(l, x)}{f(l, \varepsilon)} \; > \; 0 \right\},$$

where for integers $N \geq 1$ and $\varepsilon > 0$,

$$f(N, \varepsilon) = N^{\frac{1}{2}} (\log N)^{\frac{3}{2} + \varepsilon}.$$

We need to show that the Lebesgue measure $|E(\varepsilon)|$ of $E(\varepsilon)$ is zero for all $\varepsilon > 0$.
Note that

$$E(\varepsilon) \subseteq \bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} A_s(\varepsilon)$$

where, for a particular fixed positive constant $K > 0$,

$$A_s(\varepsilon) = \left\{ x \in [T_0, T_0 + T] : \max_{1 \leq l \leq 4^s} lD(l, x) > K^{-1} f(4^s, \frac{\varepsilon}{2}) \right\}.$$

This is because if $x \in E(\varepsilon)$ there exist arbitrarily large positive integer s such that for each integer l in $[4^{s-1}, 4^s)$ there is a real number $K > 0$, such that

$$lD(l, x) \geq f(4^{s-1}, \frac{\varepsilon}{2}) > K^{-1} f(4^s, \frac{\varepsilon}{2}).$$

In particular,

$$\max_{1 \leq l < 4^s} |lD(l, x)| > K^{-1} f(4^s, \frac{\varepsilon}{2}),$$

so $x \in E(\varepsilon)$ as required. From (*), there exist $C_2 > 0$ such that

$$|A_s(\varepsilon)| \left(f(4^s, \frac{\varepsilon}{2}) \right)^2 \leq \; C_2 4^s (\log 4^s)^2.$$

Hence, there exists $C_2 > 0$ such that

$$|A_s(\varepsilon)| \leq \frac{C_2}{s^{1+\varepsilon}}$$

and

$$\sum_{s=1}^{\infty} |A_s(\varepsilon)| \; < \; \infty,$$

so by the Borel-Cantelli Lemma, Theorem 1 is proved.

As an application to the result obtained in Theorem 1, we have the following theorem, a lighter version of Theorem 1. The result in Theorem 2 is the improvement of R.C Baker's result.

Theorem 2 *Let* $(R_k)_{k=1}^{\infty}$ *be a collection of disjoint subintervals of* $[0,1)$ *such that*

$$|R_k| = O(a^{-k}),$$

for some $a > 1$, let

$$B = \bigcup_{k=1}^{\infty} R_k.$$

Then given $\varepsilon > 0$, there exists $N_o = N_o(x, \varepsilon)$ such that if $N > N_o$

$$\left| \frac{1}{N} \sum_{n=1} \chi_B(\{\lambda_n x\}) - |B| \right| < N^{\frac{-1}{2}} (\log N)^{\frac{5}{2}+\varepsilon} \quad a.e.$$

3 Proof of Theorem 2

$$\text{For} \quad z(N) = \log_a N \qquad (N = 1, 2, ...),$$

Let

$$t(N) = \bigcup_{1 \le k \le z(N)} R_k \qquad (N = 1, 2, ...)$$

and

$$s(N) = \bigcup_{k > z(N)} R_k \qquad (N = 1, 2, ...)$$

Note that for $S \subseteq [0, 1)$, we set

$$K(S, N, x) = \frac{1}{N} \sum_{n=1} \chi_s(\{\lambda_n x\}) \; - \; |S|.$$

Then for each $l = 1, 2, ...$, the disjoinness of the R_k implies that

$$K(B, l, x) = K(t(N), l, x) + K(s(N), l, x).$$

Hence,

$$\left\| \max_{1 \le l \le N} |K(B, l, x)| \right\| \le \left\| \max_{1 \le l \le N} |K(t(N), l, x)| \right\| + \left\| \max_{1 \le l \le N} |K(s(N), l, x)| \right\|.$$

Note that by the disjointness of the R_k,

$$K(t(N),l,x) = \sum_{1\le k\le z(N)} \frac{1}{l}\left(\sum_{j=1}^{l} \chi_{R_k}(\{\lambda_j x\}) - l|R_k|\right).$$

Hence,

$$\begin{aligned}\left\|\max_{1\le l\le N}|K(t(N),l,x)|\right\| &= \left\|\max_{1\le l\le z(N)}\left|\sum_{1\le l\le z(N)}\frac{1}{l}\left(\sum_{j=1}\chi_{R_k}(\{\lambda_{R_k}x\}) - l|R_k|\right)\right|\right\| \\ &\le \sum_{1\le l\le z(N)}\left\|\max_{1\le l\le N} lD(l,x)\right\| \\ &= z(N)\left\|\max_{1\le l\le N} lD(l,x)\right\|,\end{aligned}$$

so

$$\left\|\max_{1\le l\le N}|K(t(N),l,x)|\right\| \quad \le \quad z(N)\ \left\|\max_{1\le l\le N} lD(L,x)\right\| \ ,$$

which by (*) for some $C_2 > 0$ is

$$\le\ C_2 z(N)N^{\frac{1}{2}}(\log N)\ <\ N^{\frac{1}{2}}(\log N)^2.$$

Also,

$$\begin{aligned}\left\|\max_{1\le l\le N}|K(s(N),l,x)|\right\| &= \left\|\max_{1\le l\le z(N)}\left|\sum_{1\le l\le z(N)}\frac{1}{l}\left(\sum_{j=1}\chi_{s(N)}(\{\lambda_{R_k}x\}) - l|s(N)|\right)\right|\right\| \\ &\le \sum_{j=1}^{N}\left\|\chi_{s(N)}(\{\lambda_j x\})\right\| \ +\ N|s(N)|. \qquad (**)\end{aligned}$$

As $\chi^2_{R_k} = \chi_{R_k}$, if

$$E_{k,j} = \{x\in[T_0,T_0+T\} : \{\lambda_j x\}\in R_k\}\,,$$

we see that, the right hand side of (**) is

$$\le \sum_{j=1}^{N}\left(\sum_{k>z(N)}|E_{k,j}|\right)^{\frac{1}{2}} + N|s(N)|.$$

It is easy to check that there exists $C_3 = C_3(T_0,T) > 0$ such that $|E_{k,j}| \le C_3|R_k| \ =\ O(a^{-k})$
.

Since $|s(N)| < 1$, this means that there exists $C_4 > 0$ such that

$$\left\| \max_{1 \le l \le N} |K(s(N), l, x)| \right\| \le C_4 N |s(N)|^{\frac{1}{2}}.$$

Also, there exists $C_5 > 0$ such that

$$|s(N)| = \sum_{k > z(N)} |R_k| \ \le \ C_5 \sum_{k > z(N)} a^{-k} < C_5 a^{-z(N)}.$$

Then we have,

$$\left\| \max_{1 \le l \le N} |K(B, l, x)| \right\| < N a^{-z(N)} + N^{\frac{1}{2}} (\log N)^2.$$

The first term of the right hand side is equal to 1. So we have shown that there exists $C_6 > 0$ such that

$$\left\| \max_{1 \le l \le N} |K(B, l, x)| \right\| \le C_6 N^{\frac{1}{2}} (\log N)^2.$$

Using the same argument as in Theorem 1 and choosing $f(N, \varepsilon) = N^{\frac{1}{2}} (\log N)^{\frac{5}{2} + \varepsilon}$ gives us Theorem 2.

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