

A shallow water model for the propagation of tsunami via Lattice Boltzmann method

Sara Zergani, Z. A. Aziz and K. K. Viswanathan¹

UTM Centre for Industrial and Applied Mathematics,
Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia.

Abstract. An efficient implementation of the lattice Boltzmann method (LBM) for the numerical simulation of the propagation of long ocean waves (e.g. tsunami), based on the nonlinear shallow water (NSW) wave equation is presented. The LBM is an alternative numerical procedure for the description of incompressible hydrodynamics and has the potential to serve as an efficient solver for incompressible flows in complex geometries. This work proposes the NSW equations for the irrotational surface waves in the case of complex bottom elevation. In recent time, equation involving shallow water is the current norm in modelling tsunami operations which include the propagation zone estimation. Several test-cases are presented to verify our model. Some implications to tsunami wave modelling are also discussed. Numerical results are found to be in excellent agreement with theory.

1. Introduction

General patterns and important characteristics of tsunami can be predicted by various sets of governing equations. Euler equations remain the foundation basis for tsunami propagation models, it describes the frictionless motion of water under the influence of gravity, and a number of equations that result from them in asymptotic limits '[1]', including the classical wave equation, the forced Korteweg-de Vries (fKdV) equation, the Boussinesq equation, or the shallow-water equations. This theory deals with the gravity waves problem in a regular and arbitrary depth ocean, from which tsunami wave solution is derivable from the shallow water wave approximation to its full solution. Nonlinearity plays an important part in transforming tsunami waves in the region. These difficulties can be solved by nesting near-field models like as the full Boussinesq equation model or modelling of shallow water equation with a finer mesh system to the present transoceanic model as described in '[2]'. The NSW equations are accurate for long wave propagation and runup problems, in which the scale of the vertical length scale is smaller than that of the scale of the horizontal length, such as for most tsunamis. Besides tsunami, these equations are widely used in the field of ocean engineering (e.g., tide and ocean modelling) and atmospheric modelling, where one length scale is dominant. This work focuses on the modelling of tsunami using the lattice Boltzmann method or lattice Boltzmann model for shallow water equations (LABSWE). This technique has been selected due largely to its computationally efficient basic lattice Boltzmann algorithm, and its capability to handle complex geometries and topologies. Several works have already applied LBMs to standard shallow water benchmark problems and test cases. '[3]' presented the so-called D2Q9 LBM implementation for the simulation of wave runup on a sloping beach. '[4]' applied a similar LBM to test cases including bed slope and friction terms. The main focus of their work was to demonstrate the ability of the technique to cope with multifaceted geometries and random bathymetry. Although LBMs are accepted generally to deal with complex geometries and interfacial dynamics, certain difficulties emerge in the case of boundary conditions.

2 Formulation of the problem

2.1 Governing equations

Since the scale of the vertical length is lesser than the horizontal length (refer Figure1), in order to achieve shallow water equation, both the continuity and Navier-Stokes equations have to be integrated in-depth. Shallow

¹ Corresponding author : K.K. Viswanathan, email: visu20@yahoo.com, viswanathan@utm.my



water equation are obtained from the depth integral of mass transport equation. The Coriolis effect for shallow water equations having forced wind term, bed slope and bottom friction terms is expressed as in '[5]':

$$\text{Continuity Equation: } \frac{dh}{dt} + \frac{d(hu_i)}{dx_i} = 0 \quad (1)$$

$$\text{Momentum Equations: } \frac{d(hu_i)}{dt} + \frac{d(hu_i u_j)}{dx_j} + \frac{d}{dx_i} \left(\frac{gh^2}{2} \right) = \gamma \left[\frac{\partial^2 (hu_i)}{\partial x_i \partial x_j} \right] + F_i \quad (2)$$

where i and j denote Cartesian indices and Einstein summation respectively; h is depth of the water, u_i is the i th direction-average depth velocity component, t denotes time. The term relating to forces is expressed as:

$$F_i = F_{pi} + F_{bi} + F_{wi} + F_{ci}, \quad F_{pi} = -gh \frac{\partial z_b}{\partial x_i}, \quad F_{bi} = C_b u_i \sqrt{u_i u_i}, \quad F_{wi} = \frac{\rho_a}{\rho_w} C_w u_{wi} \sqrt{u_{wi} u_{wi}}, \quad F_{ci} = \begin{cases} f_c h u_y i = x \\ -f_c h u_x i = y \end{cases},$$

where z_b is bed elevation, $g = 9.81 \text{ m/s}^2$ acceleration force due to gravity, γ viscosity of the kinematic energy and F_{pi} , is the force from outside exacted on the shallow water flow that consists of an appropriate hydrostatic pressure approximation, F_{wi} denotes wind shear stress, F_{bi} is the bed shear stress, and the Colioris effect forcing term, $F_{ci} \cdot F_c = 2\omega \sin \varphi$ is the Coriolis parameter and ω is rotation rate of the earth and φ is the latitude, z_b is the bed elevation, $C_b = g/c_z^2$ is the bed friction coefficient and $C_z = h^{1/6}/n_b$ is both the Chezy and Manning coefficients at the bed, n_b , $\left[\frac{1}{L^3 T} \right] \cdot \rho_w$ is the water density, ρ_a the air density, $C_w = (0.63 + 0.66 \sqrt{u_{wi} u_{wi}} \times 10^{-3})$ is the expression for the coefficient of the wind, and u_{wi} is the i th direction velocity of wind.

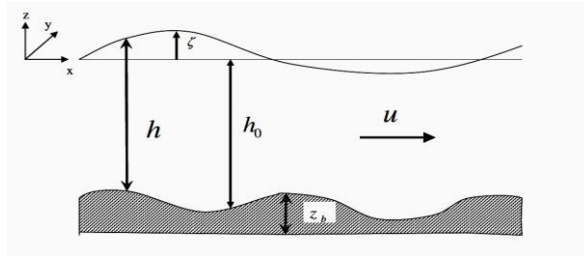


Figure 1 Shallow water flow regime

2.2 Lattice Boltzmann equation

The LBM is a numerical method for solution of flow equations without using the complicated shallow-water equations. It solves the lattice Boltzmann equation, and the depth and velocity can be calculated from macroscopic properties. Only simple arithmetic calculations are required to generate accurate solutions to flow problems with straightforward treatment of boundary conditions, and providing an easy and efficient way to simulate complicated flows. The lattice Boltzmann equation which includes a force term on a nine-velocity square lattice is given by

$$f_\alpha(X + e_\alpha \Delta t, t + \Delta t) - f_\alpha(X, t) = -\frac{1}{\tau} (f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{6e^2} e_{\alpha i} F_i \quad (3)$$

where $f_\alpha(x, t)$ is particle distribution function, x is space vector, t is time, e_α is particle velocity vector, where $\alpha = 1, \dots, 9$, $e = \Delta x / \Delta t$, Δx is lattice size, Δt is time step, τ is single relaxation time factor. The stability of the equation requires that $\tau > 1/2$ and $f_\alpha^{(eq)}(x, t)$ is the equilibrium distribution function at time t . F_i is the i th direction force component.

2.3 Definition of macroscopic quantity

$$\text{The water depth } h \text{ is given as } h(X, t) = \sum_\alpha f_\alpha(X, t) = \sum_\alpha f_\alpha^{eq}(X, t) \quad (4)$$

The macroscopic quantity velocity $u(X, t)$ is defined as

$$u_i(X, t) = \frac{1}{h(X, t)} \sum_{\alpha} e_{\alpha i} f_{\alpha}(X, t) = \frac{1}{h(X, t)} \sum_{\alpha} e_{\alpha i} f_{\alpha}^{eq}(X, t) \quad (5)$$

$$\sum_{\alpha} e_{\alpha i} e_{\alpha j} f_{\alpha}^{eq}(X, t) = \frac{1}{2} g h^2(X, t) \delta_{ij} + h(X, t) u_i(X, t) u_j(X, t) \quad (6)$$

2.4 Equilibrium Distribution Function For 2D Shallow Water Equations

Considering the theory of the lattice gas automata, the equilibrium function is the Maxwell-Boltzmann equilibrium distribution function. This distribution function is often expanded as a Taylor series in macroscopic velocity to its second order. It is assumed that an equilibrium function can be stated as a power series in macroscopic velocity

$$f_i^{eq} = A_{\alpha} + B_{\alpha} e_{\alpha i} u_i + C_{\alpha} e_{\alpha i} e_{\alpha j} u_i u_j + D_{\alpha} u_i u_j, \quad (7)$$

It is convenient to write the equation above in the following form,

$$f_{\alpha}^{eq} = \begin{cases} A_0 + D_0 u_i u_i, & \alpha = 0 \\ A + B e_{\alpha i} + C e_{\alpha i} e_{\alpha j} u_i u_j + D u_i u_i, & \alpha = 1, 3, 5, 7 \\ A + B e_{\alpha i} u_i + C e_{\alpha i} e_{\alpha j} u_i u_j + D u_i u_j, & \alpha = 2, 4, 6, 8 \end{cases} \quad (8)$$

The coefficients can be determined based on the limitations of the equilibrium distribution function. The macroscopic quantity's three conditions must be satisfied by the local equilibrium distribution function in shallow water equation. The calculation of the Lattice Boltzmann equation leads to the resolution of the 2D equation for shallow water if the local equilibrium function could be determined under the above constraint. Substituting 'equation (8)' in 'equations (4-6)' and evaluating the terms with

$$e_{\alpha} = \begin{cases} (0, 0), & \alpha = 0 \\ e \left[\cos \frac{(\alpha-1)\pi}{4}, \sin \frac{(\alpha-1)\pi}{4} \right], & \alpha = 1, 3, 5, 7 \\ \sqrt{2} \left[\cos \frac{(\alpha-1)\pi}{4}, \sin \frac{(\alpha-1)\pi}{4} \right], & \alpha = 2, 4, 6, 8 \end{cases} \quad (9)$$

and after the coefficients are decided, this results in

$$f_{\alpha}^{eq} = \begin{cases} h - \frac{5gh^2}{6e^2} - \frac{2hu_i u_i}{3e^2}, & \alpha = 0 \\ \frac{gh^2}{6e^2} + \frac{he_{\alpha i} u_i}{3e^2} + \frac{he_{\alpha i} e_{\alpha j} u_i u_j}{2e^4} - \frac{hu_i u_i}{6e^2}, & \alpha = 1, 3, 5, 7 \\ \frac{gh^2}{24e^2} + \frac{he_{\alpha i} u_i}{12e^2} + \frac{he_{\alpha i} e_{\alpha j} u_i u_j}{8e^4} - \frac{hu_i u_i}{24e^2}, & \alpha = 2, 4, 6, 8 \end{cases} \quad (10)$$

2.5 Boundary and initial conditions

In order to solve shallow water flow problems by use of LABSWE, suitable boundary conditions must be provided. Generally speaking, in the application of boundary conditions in LBM, the temporal/spatial flexibility is allowed. This is briefly described as follows: solid boundary conditions; no-slip or slip boundary conditions may be used. For no-slip conditions, the normal bounce-back scheme can be applied. For slip conditions, a zero gradient of the distribution function perpendicular to the solid wall can be employed. Representation of boundary inflow and outflow and periodic boundary conditions are used in the verification of the models.

3 Results and Discussion

In the following, the NSW-LBM code is applied to three benchmark problems or test-cases widely used in the tsunami community: (i) 2D Tidal flow over a regular bed in 3D plot; and (ii) 2D Tidal flow over an irregular bed in 3D plot; and (iii) 2D steady flow over an irregular bed in 3D plot. In all cases, the LBM solution is

compared to the available analytical/numerical or experimental reference solutions, and to other results from the literature.

3.1 2D Tidal flow over a regular bed in 3D plot

Tidal waves often occur in coastal engineering. ‘[6]’ test problem is used in verifying upwind discretisation of the source of bed slope terms. A two dimensional problem in which is defined the bed topography as $G(x) = 50.5 - 40x/L - 10\sin[\pi(4t/L - 1/2)]$, where $G(x)$ is the incomplete depth between a preset reference plane and the bed plane, giving $Z_b(x) = G(0) - G(x)$. The initial water height and velocity are $g(x,0) = G(x)$ and $u(x,0) = 0$. At the channel’s inflow and outflow, we define $g(0,t) = 20 - 4\sin\left[\pi\left(\frac{4t}{86400} + \frac{1}{2}\right)\right]$ and $u(L,t) = 0$ respectively. The asymptotic analytical solution for the short tidal wave is given by ‘[6]’:

$$g(x,t) = G(x) + 4 - 4\sin\left[\pi\left(\frac{4t}{86400} + \frac{1}{2}\right)\right], \quad u(x,t) = \frac{(x-14000)}{5400g(x,t)} \cos\left[\pi\left(\frac{4t}{86400} + \frac{1}{2}\right)\right] \quad (11)$$

The D2Q9 velocity model is used. The slip or non-slip boundary conditions are used at the solid walls. For the non-slip condition, the bounce-back plan is used and for slip conditions, a zero gradient of the distribution function perpendicular to the solid wall is employed and periodic boundary conditions are applied in the upper and lower walls. To achieve a lattice-independent solution, lattices of 15000×6000 and the lattice speed $e = 200m/s$ and $\tau = 0.6$ are also used. The results are given in ‘figures 2-5’. This confirms the accuracy of the model for unsteady shallow-water flow problems. The present method can provide solution of equal accuracy as found in ‘[6]’, where they used a complex upwind discretisation for the source term of the bed slope.

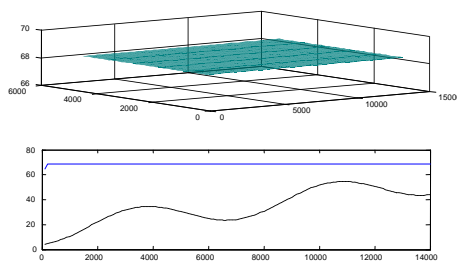


Figure 2 Numerical tidal free surface wave flow at time $t=0.0$

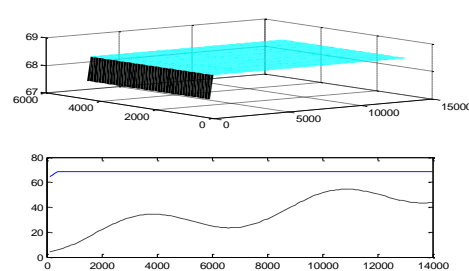


Figure 3 Numerical tidal free surface wave flow at time $t=1.00$.

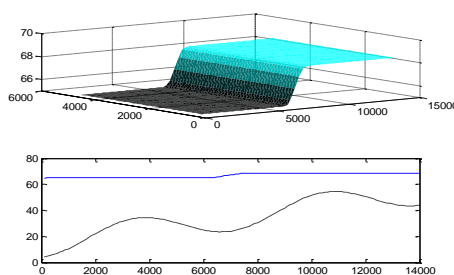


Figure 4 Numerical tidal free surface wave flow at time $t=50.0$.

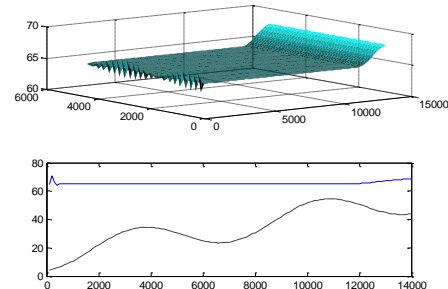


Figure 5 Numerical tidal free surface wave flow at time $t=100.00$.

3.2 2D Tidal flow over an irregular bed in 3D plot

Let us consider a tidal flow that occurs over a bed that is known not to be regular as further test for the capability of the LBM for shallow-water equation. The bed is the same as that defined in ‘Table 1’. For numerical computations, the D2Q9 velocity model is used with f^{eq} defined. The structure of the grid contains 150×60

lattice points with, the initial and boundary conditions are $g(x,0)=16-Z_b(x)$ and $u(x,0)=0$ and $g(0,t)=20-4\sin\left[\pi\left(\frac{4t}{86400}+\frac{1}{2}\right)\right]$ and $u(L,t)=0$, and thus the asymptotic analytical solution of the short tidal flow is

$$g(x,t)=20-Z_b(x)-4\sin\left[\pi\left(\frac{4t}{86,400}+\frac{1}{2}\right)\right], \quad u(x,t)=\frac{(x-L)\pi}{5,400g(x,t)}\cos\left[\pi\left(\frac{4t}{86,400}+\frac{1}{2}\right)\right] \quad (12)$$

The centred scheme is employed for the force term. In order to contrast the numerical results and the asymptotic analytical solution, we select two results at $t=10,800s$ and $t=32,400s$, which relate to the half-risen tidal flow with maximum positive velocities and to the half-ebb tidal flow with maximum negative velocities and presented the results as shown in ‘figures 6-8’. These figures shows that the numerical calculations and that obtained analytically are excellently in agreement. This supports the claim that the centred scheme is likewise correct and conservative for tidal flow that occurs over a bed that is known to be irregular. The results acquired with the basic and second order schemes are also carried out. Comparisons are done for the water surface and maximum positive velocities at $t=10,800s$. It is obvious from the figures that only the centred scheme yield accurate result.

Table1 Bed elevation z_b at point x for irregular bed.

x	0	50	100	150	250	300	350	400	425	435	450	475	500	505
z_b	0	0	2.5	5	5	3	5	5	7.5	8	9	9	9.1	9
x	530	550	565	575	600	650	700	750	800	820	900	950	1000	1500
z_b	9	6	5.5	5.5	5	4	3	3	2.3	2	1.2	0.4	0	0

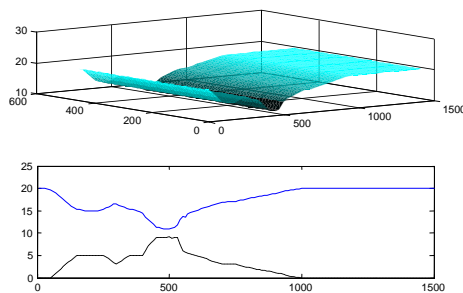


Figure 6 Numerical free surface for tidal flow over an irregular bed at time $t = 1.0$.

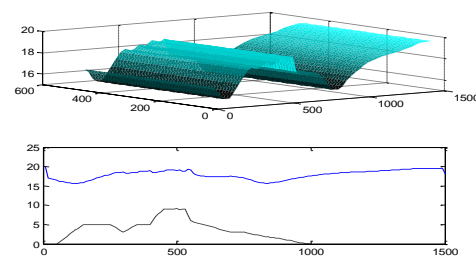


Figure 7 Numerical free surface for tidal flow over an irregular bed at time $t = 30.0$.

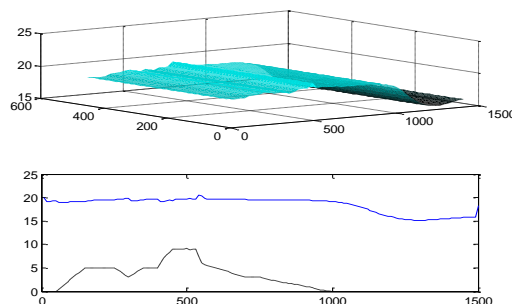


Figure 8 Numerical free surface for tidal flow over an irregular bed at time $t = 70.00$.

3.3 2D steady flow over an irregular bed in 3D plot

The bed landscape is described in ‘tables 1-3’ are shown in ‘figures 9-12’.

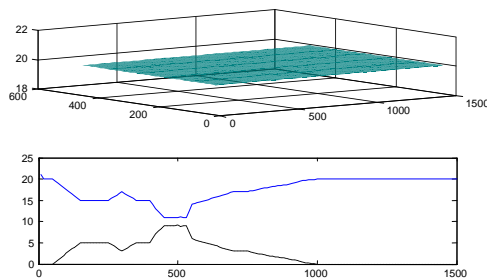


Figure 9 Numerical free surface for steady flow over an irregular bed at time $t = 1.00$

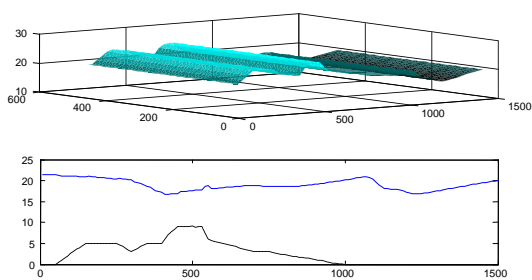


Figure 10 Numerical free surface for steady flow over an irregular bed at time $t = 2.00$

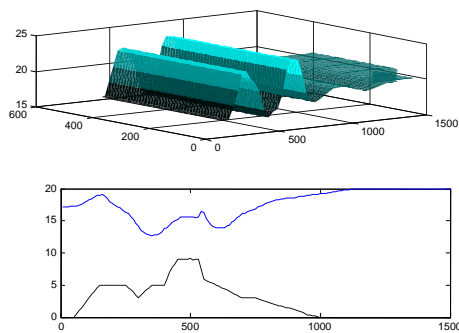


Figure 11 Numerical free surface for steady flow over an irregular bed at time $t = 10.00$.

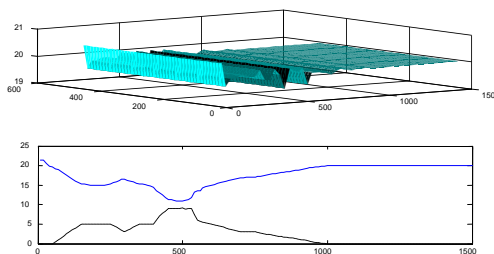


Figure 12 Numerical free surface for steady flow over an irregular bed at time $t = 60.00$

Table 2. Values of various parameters used for the wave Propagation over an oscillatory bottom test-case.

Initial wave number k	1 m^{-2}
Gravity acceleration g	$1 \times ds \text{ ms}^{-2}$
Initial wave amplitude b	0.2 m
Undisturbed water depth d_0	1 m
Bathymetry oscillation amplitude α	0.001 m
Low bathymetry oscillation wavelength k_2	2 m^{-2}
High bathymetry oscillation wavelength k_z	6 m^{-2}
relaxation time τ	1
Real Channel Length rx	20

Table3. Values of various parameters used for different test-case.

Values of various parameters used	Tidal Flow over a Regular Bedtest-case.	Tidal Flow over an Irregular Bedtest-case	Steady Flow over a Bumptest-case	Steady Flow over an Irregular Bedtest-case
relaxation time τ	0.6	1.5	1.1	0.9
Real Channel Length rx	14000m	1500m	2m	1500m
dx	0.05m	0.05m	0.05m	0.05m
h_0	64.5m	20m	1.8m	20m
Gravity acceleration g	$9.81 \times dx$	$9.81 \times dx$	$9.81 \times dx$	$9.81 \times dx$
u_x	0	0	0.005	0.05
u_y	0	0	0	0

4 Conclusions

A new lattice Boltzmann model is suggested to solve the 2D NSW wave equations. The efficiency and accuracy of the model are confirmed via thorough numerical simulation with lattice Boltzmann equation. It is noted that in order to attain better accuracy the LABSWE requires a relatively small time step Δt and the proper range is from 10^{-3} to 10^{-4} . The work would like to underline the importance of a robust runup algorithm development using the current model. This research should shift forward the accuracy and comprehension of a water wave runup onto complex shores. The results obtained reveal that NSW equation has sufficient prediction ability for maximum runup value. In conclusion, we have used three examples to test the LBM. It can be concluded that LBM performs well for such problems. The numerical results agree with the theory and hence, one can conclude that the stability structure is a good tool for designing the LBM. Furthermore, on the time-dependent problems on the unsteady problem, excellent and accurate results are obtained with no additional steps on the source terms or complicated upwind discretization of the gradient fluxes.

5 Acknowledgments

This work is supported by MOHE, Flagship Project Vote No. 01G40 under Research Management Center (RMC), Universiti Teknologi Malaysia, Johor Bahru, Malaysia.

REFERENCES

- [1] Yaacob N, Aziz Z A and Norhafizah M S 2008 Modelling of Tsunami waves. *Matematika*. **24** 211-230.
- [2] Yoon T H and Kang S K 2004 Finite Volume Model for Two-Dimensional Shallow Water Flows on Unstructured Grids. *Journal of Hydraulic Engineering*. **130** 678-688.
- [3] Frandsen J B 2006 Investigations of wave runup using a LBGK modeling approach. In *CMWR XVI Proc. XVI Intl. Conf. Comp.Meth.in Water Resoures*, Copenhagen, Denmark.
- [4] Thömmes G, Seaid M and Banda M K 2007 Lattice Boltzmann methods for shallow water flow applications. *Int. J. Numer. Meth.Fluids*. **55** 673–692.
- [5] Zhou J G 2002 A lattice Boltzmann model for the shallow water equations. *Computer Methods in Applied Mechanics and Engineering*. **191** 3527-3539.
- [6] Bermudez A and Vázquez M E 1994 Upwind methods for hyperbolic conservation laws with source Terms. *Comput. Fluids*. **23**, 1049.