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TAJUK PROJEK :

*Stiff PDE in Heat Problem :  
Solution Using the Method of lines  
with New Numerical Algorithm*

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**STIFF PDE IN HEAT PROBLEM : SOLUTION  
USING THE METHOD OF LINES WITH NEW  
NUMERICAL ALGORITHM**

**ASSOC. PROF. DR. NAZEERUDDIN YAACOB**

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**Solving Stiff Equations in Heat Problem Using Five-points Method  
of Lines and New Third Order Runge-Kutta Method**

**By**

**Nazeeruddin Yaacob**

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## Abstract

An equation where solutions change on two vastly different scales will encounter a stiff problem. Partial differential equations can lead to systems of first order ordinary differential equations when discretized using finite difference such as methods of lines. The method of lines, (MOL) is a powerful technique for solving partial differential equation. This project aims to demonstrate the combination of two methods in order to solve the stiff problems. The methods are the method of lines with five-points central finite difference and the explicit third order Runge-Kutta method.

## Abstrak

Suatu persamaan dimana penyelesaian berubah besar pada dua skala berbeza akan menemui masalah kekakuan. Persamaan pembeza separa boleh menjadi sistem persamaan pembeza biasa peringkat pertama apabila di diskritkan dengan kaedah beza terhingga seperti kaedah garisan. Kaedah garisan merupakan satu kaedah yang sangat berkesan untuk menyelesaikan persamaan pembeza separa. Projek ini bertujuan memperlihatkan kombinasi dua kaedah ini untuk menyelesaikan masalah kekakuan. Kaedah-kaedah tersebut adalah kaedah garisan dengan lima-titik beza terhingga dan kaedah tak tersirat runge-Kutta peringkat ketiga.

## 1. Introduction

A stiff system is one having one or more fast decay processes along with relatively slow processes, such that the shortest decay “time constant” is much smaller than the total span of interest in the independent variable, which is usually “time” [1]. Systems of ordinary differential equations arise frequently, in almost every discipline of science and engineering, as a result of modelling and simulation activities. Large stiff ordinary differential equations initial value problems occur in a fairly direct manner in the modelling of electrical networks, mechanical system, chemical reactors, biological system and nuclear reactors, to name just a few of many. A less direct but extremely common source of large stiff systems is that time-dependent partial differential equations, after a discretization process in the spatial variable(s). Three categories of partial differential equations discretizations which are normally used: finite difference method, finite element methods and spectral methods, which involve the Fourier series.

In this project the method of lines is applied to the parabolic partial differential equation problem, which we use as our example, to reduce it to a simpler ordinary differential equation problem by discretizing all except one of the independent variables. The parabolic partial differential equation, which is a diffusion type equation, also known as the heat equation, produced a system of ordinary differential equation which, in this particular example, consists of four first order ordinary differential equations. This first order ordinary differential equations system is then represented in the matrix form and we calculated the eigenvalues of the matrix to identify the character of the system. Referring to the definition of stiffness by Lambert, a differential equation is stiff if some or all of these eigenvalues have a real

part which is negative and of large magnitude. We found that we have a stiff system since the calculation attained large magnitude and negative eigenvalues.

The purpose of this project is to present efficient general purpose methods for the numerical solution of stiff partial differential equations. In particular, we hope that this method will not be any more time-consuming compared to the standard technique such as fourth order Runge-Kutta method.

## 2. Literature Review

Many fields of application, notably the engineering mathematical models that concerned with the partial differential equations, yield initial value problems involving systems of ordinary differential equations. These problems exhibit a phenomenon, which is known as 'stiffness'. A stiff differential equations is one with general solution contains an exponential term such as  $e^{\lambda x}$  for some constant  $\lambda$ . If  $\lambda$  is a large negative quantity, such equations are particularly troublesome since it causes the solution to decay rapidly to zero. Curtiss and Hirschfelder did the research of this area since 1952. Anyway, Dahquilst was the person who brings the problem to the attention of the numerical computing company. A first survey on method for stiff problems by Bjurel et al (1970) reports over forty methods, which most of them are implicit ones. In 1996, T. Van Hecke *et al* had developed a variable step order algorithm for stiff ODE's.

Again in 1996, Butcher, Cash and Diamantakis introduced the diagonally extended singly implicit (DESI) Runge-Kutta methods for stiff initial value problems to overcome some of the limitations of the singly implicit methods. These methods experimentally have shown its competitiveness with the backward differential

formulae methods [2]. In short, the research in this area is still on going and has attracted many scholars.

Most of the research focuses on the accuracy and stability of the method.

This research will make use of the method of lines, which is so far the most flexible method in spatial discretization in order to transform the PDEs into systems of ODEs.

The systems of ODEs obtained are usually stiff and highly expensive to solve. The new formula of third-order Runge-Kutta method, which we developed, will be used to solve the systems of ODEs. The advantage of this new formula is that it is an explicit method, which we hope will lower the cost of computations.

### 3. Methodology

We used method of lines in our project. This methods (MOL) has been used traditionally to solve partial differential equations (PDE). The basic idea of this particular method is to transform the PDE to simpler ordinary differential equations by discretizing all except one of the independent variables. When there are two independent variables, one is discretized while the other is left continuous. Once the PDE problems have been approximated by simpler ODE, then any suitable numerical methods which deal with stiff problems can be used to obtained an approximate solution.

In this project, we use the five-point central differences in order to semidiscretize the PDEs. The formula is given by [3]

$$u'_j(t) = \frac{-u_{j+2}(t) + 16u_{j+1}(t) - 30u_j(t) + 16u_{j-1}(t) - u_{j-2}(t)}{12h^2}$$



The new third order Runge-Kutta method is developed in this project is combined with the MOL in the attempt to solve the stiff problem arises in heat problem.

$$\begin{aligned}k_1 &= f(x, y_n) \\k_2 &= f(x + c_1 h, y_n + a_1 h k_1) \\k_3 &= f(x + c_2 h, y_n + a_2 h k_1 + a_3 h k_2) \\y_{n+1} &= y_n + \frac{h}{4} (b_1 k_1 + b_2 k_2 + b_3 k_3)\end{aligned}$$

$$\text{where } c_1 = \frac{2-\sqrt{2}}{3}, c_2 = \frac{1}{3}, a_1 = \frac{2-\sqrt{2}}{3}, a_2 = a_3 = \frac{1}{6}, b_1 = 4 + 3\sqrt{2}, b_2 = -3(4 + 3\sqrt{2}) \\ \text{and } b_3 = 6(2 + \sqrt{2})$$

This formula will be enhanced for the system of first order differential equations.

#### 4. Numerical Results

We solve the following heat equation

$$U_t = \alpha^2 U_{xx}; \quad 0 < x < 1m$$

with initial condition

$$U(x, 0) = 70^\circ C, \quad 0 \leq x \leq 1m$$

and boundary condition

$$U(0, t) = 50^\circ C, \quad 0 < t < 0.3hr$$

$$U(1, t) = 70^\circ C.$$

where  $\alpha^2 = 0.1$  and  $\Delta x = 0.2m$

The problem can be written as a system of first order ODEs

$$\begin{aligned}\frac{du_1}{dt} &= \frac{5}{24}(-u_{-1} + 16u_0 - 30u_1 + 16u_2 - u_3) \\ \frac{du_1}{dt} &= \frac{5}{24}(-u_0 + 16u_1 - 30u_2 + 16u_3 - u_4) \\ \frac{du_1}{dt} &= \frac{5}{24}(-u_1 + 16u_1 - 30u_2 + 16u_3 - u_4) \\ \frac{du_1}{dt} &= \frac{5}{24}(-u_2 + 16u_3 - 30u_4 + 16u_5 - u_6)\end{aligned}$$

with  $u_0 = 50^\circ C$  and  $u_5 = 20^\circ C$

In dealing with the five-point central difference approximations to the second derivative Fisher [ ], proposed an assumption which leads to

$$u_{n+1} = -u_{-(n-1)} \text{ and } u_{-(n+1)} = -u_{(n+1)}$$

Combining the assumption and the average of steady state boundary and initial value, we obtained  $u_{-1} = -60^\circ C$  and  $u_6 = -45^\circ C$ .

In matrix form, the above system becomes

$$\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \\ \frac{du_3}{dt} \\ \frac{du_4}{dt} \end{pmatrix} = \begin{pmatrix} -30 & 16 & -1 & 0 \\ 16 & -30 & 16 & -1 \\ -1 & 16 & -30 & 16 \\ 0 & -1 & 16 & -30 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

The eigenvalues for the matrix above are

$$\lambda_1 = -39, \lambda_2 = -5, \lambda_3 = -38 - \sqrt{353}, \text{ and } \lambda_4 = -38 + \sqrt{353}$$

The negative eigenvalues above show that we have a stiff system of ODEs. Using the new third order Runge-Kutta method (namely SAM) to solve the above system we obtained the following results in Table 1

Table 1. The relative errors for the 5-point MOLs + SAM method on the parabolic partial differential equations compared to the forward difference method.

$x = 0.2$	Exact value	Relative errors	
		Forward Difference	5-point MOLs + SAM
$t = 0.06$	68.6422168761	2.3924298352E-02	8.9805577352E-03
$t = 0.12$	66.0658758630	1.7647171823E-02	8.9377506713E-03
$t = 0.18$	64.1619477795	1.2459842744E-02	2.3324865269E-02
$t = 0.24$	62.7607571081	9.4696054587E-03	3.3742188556E-02
$t = 0.30$	61.6611828442	7.9046139194E-03	4.1507012244E-02
$x = 0.4$			
$t = 0.06$	69.9947833016	7.4529817137E-05	9.9740365315E-04
$t = 0.12$	69.7981589510	3.5553796079E-03	5.8912909727E-05
$t = 0.18$	69.2214302667	6.7924090110E-03	2.5464532589E-04
$t = 0.24$	68.3337219450	9.0525867375E-03	2.0899111914E-03
$t = 0.30$	67.2352979293	1.0437799428E-02	5.0889971620E-03
$x = 0.6$			
$t = 0.06$	69.9869625901	1.8628340790E-04	1.1563680729E-03
$t = 0.12$	69.5066860511	9.0881336313E-03	9.7162224015E-03
$t = 0.18$	68.2179429039	1.4445508937E-02	1.3218897889E-02
$t = 0.24$	66.4821442738	1.6482159018E-02	1.2660850260E-02
$t = 0.30$	64.5903609103	1.6910296597E-02	9.9623064334E-03
$x = 0.8$			
$t = 0.06$	66.6055422721	6.1639649375E-02	5.6624039992E-02
$t = 0.12$	60.1647150419	4.8445588745E-02	3.9084994391E-02
$t = 0.18$	55.4074762158	3.6118342731E-02	2.2660337205E-02
$t = 0.24$	51.9292648451	2.8115453771E-02	1.0620207231E-02
$t = 0.30$	49.2673936515	2.2955596342E-02	1.3639569719E-03

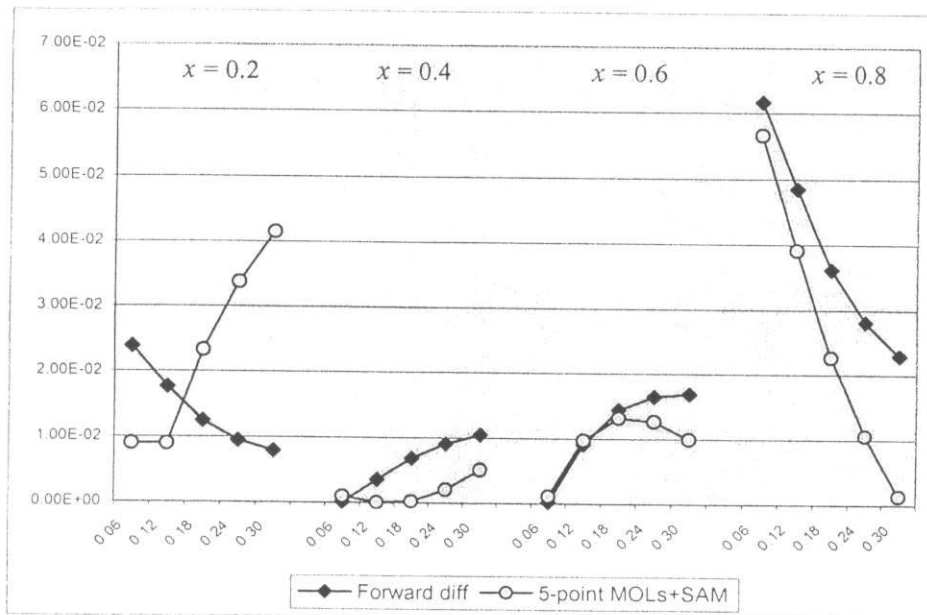


Figure 1: The graphs relative errors obtained using the 5-point MOLs + SAM and the forward difference methods at each point of  $x$ .

We illustrate the results of the temperature achieved at each point using the two methods mentioned above together with the exact solution derived analytically.

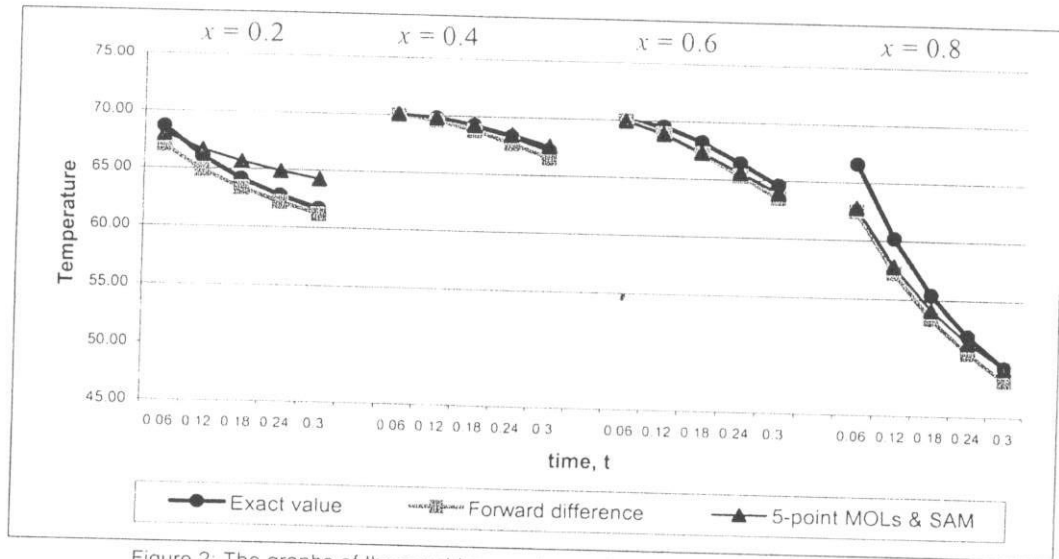


Figure 2: The graphs of the exact temperature and temperatures obtained using the 5-point MOLs + SAM and the forward difference methods at each point of  $x$ .

## 5. Discussion & Conclusion

This research deals with two methods which are combined to combat stiffness occurred in the partial differential equation. The first method is the method of lines, which we use to transform the partial differential equation into system of ordinary differential equation. The second method is the explicit third-order stage-arithmetic mean Runge-Kutta method, which is applied in the final step in completing the task to crack the stiff problem. We also solve the same problem using the forward difference method, which as a result, we achieved a better accuracy in the new combined-method compared to the other method. The graphs and values of the relative errors are exhibited to highlight

the comparison of both methods. We also plot the graphs of the exact temperatures and the temperatures achieved using both methods at each point,  $x$ .

We find that the combination of the 5-point MOLs and ~~stage-arithmetic mean~~ <sup>new third order</sup> Runge-Kutta method “loses” a few points at the  $x = 0.2$  but “gains” back the accuracy at the other points. We had a feeling that the combination of assumptions for the endpoints of the 5-point MOLs could still be improved in order to gain better accuracy.

We conclude that as a whole, this new creation of combination of 5-point MOL and the ~~stage-arithmetic mean~~ <sup>new third order</sup> Runge-Kutta method works better compared to the forward difference method.

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## Third-order composite Runge–Kutta method for stiff problems

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The derivation of a composite method for solving stiff ordinary differential equations is discussed. Combination of the harmonic and arithmetic means of the Runge–Kutta formulation has resulted in the introduction of a new formula for the numerical solution of stiff ordinary differential equations. The numerical results and the A-stability of this new formula are examined.

**Keywords:** Runge–Kutta; Stiff ODEs; Arithmetic mean; Harmonic mean; A-stability

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### 1. Introduction

The problem of stiffness has been known for some time and has attracted the attention of many numerical analysts, leading to surveys of methods for stiff problems. A considerable amount of research in recent years has been directed to the problem of solving stiff systems by using implicit methods [1]. It is also believed that implicit Runge–Kutta methods are appropriate for the solution of stiff problems [2]. It has been suggested that stiff equations cannot be solved using explicit methods, but recently a sixth-order A-stable explicit one-step method has been developed using a composite of a polynomial and an exponential function [3]. In this paper, we show that an Runge–Kutta-like explicit method can also be used to solve stiff problems.

It was shown in [4] that the harmonic mean (HM) Runge–Kutta formula can be written in the form

$$y_{n+1} = y_n + \frac{h}{2} \left( \frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3} \right) \quad (1)$$

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where

$$\begin{aligned}k_1 &= f(x_n, y_n) = f \\k_2 &= f(x_n + a_1 h, y_n + a_1 h k_1) \\k_3 &= f(x_n + (a_2 + a_3)h, y_n + h a_2 k_1 + h a_3 k_2).\end{aligned}$$

Third-order accuracy was obtained by adjusting the parameters through the solution of equations of condition:

$$\begin{aligned}2a_1 + a_2 + a_3 &= 2 \\12a_1 a_3 + 9a_1 a_2 + 3a_1^2 &= 8 \\3(a_2 + a_3)^2 + 6a_1^2 &= 4\end{aligned}\tag{2}$$

to obtain,  $a_1 = 2/3$ ,  $a_2 = -2/3$ ,  $a_3 = 4/3$ .

## 2. Formulation of the third-order composite of the arithmetic and harmonic means

It is possible to form a composite of the arithmetic and harmonic means, using one in the main formula and the other in the stages. Using the concept of the arithmetic mean in the stages, we have

$$\begin{aligned}k_1 &= f(x_n, y_n) = f \\k_2 &= f(x_n + a_2 h, y_n + a_2 h k_1) \\k_3 &= f\left(x_n + a_3 h, y_n + a_3 h \left(\frac{k_1 + k_2}{2}\right)\right) \\y_{n+1} &= y_n + \frac{h}{2} \left(\frac{2k_1 k_2}{k_1 + k_2} + \frac{2k_2 k_3}{k_2 + k_3}\right).\end{aligned}\tag{3}$$

Since the algebra involved is the division of two series

$$\frac{2k_i k_{i+1}}{k_i + k_{i+1}}, \quad i = 1, 2, 3\tag{4}$$

we cannot make a direct substitution. These problems are alleviated by cross-multiplying the terms by the common denominator  $(k_1 + k_2)(k_2 + k_3)$ , which can be written as

$$y_{n+1} = y_n + \frac{\text{TOP}}{\text{BOTTOM}}\tag{5}$$

where

$$\begin{aligned}\text{TOP} &= h[k_2 k_3 (k_1 + k_2) + k_1 k_2 (k_2 + k_3)] \\ \text{BOTTOM} &= (k_1 + k_2)(k_2 + k_3).\end{aligned}$$

Since the error of the method can be determined using the expression

$$\text{error} = y(x_{n+1}) - y_{n+1} = y(x_{n+1}) - y_n - \frac{\text{TOP}}{\text{BOTTOM}}$$

we obtain

$$\text{error} = \text{Taylor} - \frac{\text{TOP}}{\text{BOTTOM}}$$

which can be written as

$$\text{error} \times \text{BOTTOM} = \text{Taylor} \times \text{BOTTOM} - \text{TOP}.\tag{6}$$

We compared the coefficients of the same terms in (6) up to the term  $h^3$  and, using Mathematica [5], we obtained two sets of parameters:

$$a_2 = 0, \quad a_3 = 2$$

and

$$a_2 = \frac{3}{5}, \quad a_3 = \frac{4}{5}.$$

The composite arithmetic-harmonic mean Runge-Kutta formula can be represented as follows.

*Set 1:*

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n, y_n) = k_1 \\ k_3 &= f\left(x_n + 2h, y_n + 2h\left(\frac{k_1 + k_2}{2}\right)\right) \\ &= f(x_n + 2h, y_n + 2hk_1) \\ y_{n+1} &= y_n + \frac{h}{2} \left( \frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3} \right) \\ &= y_n + \frac{h}{2} \left( k_1 + \frac{2k_1k_3}{k_1 + k_3} \right) \end{aligned} \quad (7)$$

since  $k_2 = k_1$ .

*Set 2:*

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f\left(x_n + \frac{3}{5}h, y_n + \frac{3}{5}hk_1\right) \\ k_3 &= f\left(x_n + \frac{4}{5}h, y_n + \frac{4}{5}h\left(\frac{k_1 + k_2}{2}\right)\right) \\ y_{n+1} &= y_n + \frac{h}{2} \left( \frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3} \right). \end{aligned} \quad (8)$$

In order to obtain the 'best' result, we used both sets of coefficients given above to solve stiff ordinary differential equations. The 'best' accuracy was achieved using the coefficients in set 2.

### 3. Stability

To check the stability, we substitute formula (8) of set 2 in the simple test equation

$$y' = \lambda y.$$

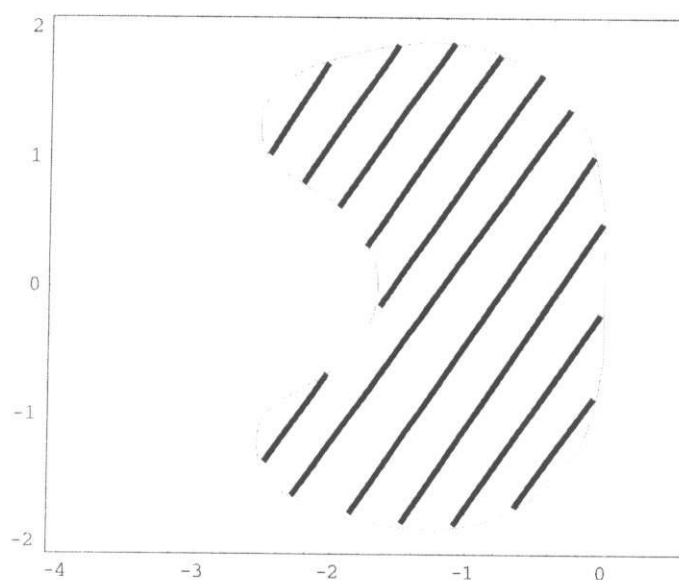


Figure 1. The 2D stability region plotted for the composite method.

and we have

$$y_{n+1} = y_n + \frac{h}{2} \left( \frac{2\lambda y_n(1 + 3\lambda h/5)}{\lambda y_n + \lambda y_n(1 + 3\lambda h/5)} + \frac{2\lambda y_n(1 + 3\lambda h/5)\lambda y_n\{1 + [4\lambda h(2 + 3\lambda h/5)]/10\}}{\lambda y_n(1 + 3\lambda h/5) + \lambda y_n\{1 + [4\lambda h(2 + 3\lambda h/5)]/10\}} \right)$$

Setting  $z = \lambda h$ , we obtain the simplified equation

$$y_{n+1} = y_n \left\{ 1 + \frac{z(5 + 3z)}{10 + 3z} \left[ \frac{50 + 30z + 6z^2}{5(5 + 2z)} \right] \right\} \quad (9)$$

and

$$R(z) = 1 + \frac{z(5 + 3z)}{(10 + 3z)} \left[ \frac{50 + 30z + 6z^2}{5(5 + 2z)} \right] \quad z \neq -2.5, -10/3.$$

The stability regions for the above formula are illustrated in figures 1 and 2.

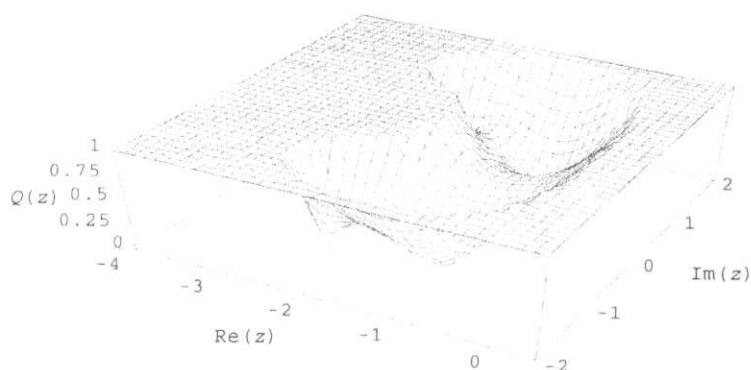


Figure 2. The 3D stability region plotted for the composite method.

#### 4. Numerical example

We consider the initial value problem

$$f(t, y) = -100y(t) + e^{-2t}; \quad y(0) = 0$$

with the exact solution

$$y(t) = \frac{1}{98}e^{-100t}(-1 + e^{98t})$$

over the range  $0 \leq t \leq 1.0$ .

We solve a stiff initial value problem equation using set 2 to obtain the 'best' results. Using a step size  $h = 0.01$ , we printed the numerical results obtained every 10 steps using equation (8). For comparison, the numerical solution and the relative errors obtained using the modified third-order mean Runge-Kutta method [6] are shown.

Table 1 shows the exact values and the relative errors for each method, and figure 3 shows the errors diagrammatically.

Table 1. Errors in the composite Arithmetic-Harmonic Runge-Kutta method compared to the modified third order method (Wazwaz)

$t$	Exact value	GM <sup>2</sup> /AM (Wazwaz)	Relative error 1	Arith-harmonic	Relative error 2
0.1	0.0083539322	0.0080357102	3.8092481512E-02	0.0083687919	1.77876430833E-03 ✓
0.2	0.0068400004	0.0065740809	3.8877129698E-02	0.0068404782	6.98463053522E-05 ✓
0.3	0.0056001187	0.0053819215	3.8962962029E-02	0.0056005066	6.92519801110E-05
0.4	0.0045849894	0.0044063005	3.8972599992E-02	0.0045853069	6.92517715086E-05
0.5	0.0037538718	0.0036075697	3.8973674574E-02	0.0037541318	6.92517714399E-05
0.6	0.0030734103	0.0029536279	3.8973794285E-02	0.0030736232	6.92517714360E-05
0.7	0.0025162956	0.0024182259	3.8973807620E-02	0.0025164698	6.92517714306E-05
0.8	0.0020601686 ✓	0.0019798759	3.8973809105E-02	0.0020603112 ✓	6.92517714307E-05
0.9	0.0016867233	0.0016209853	3.8973809271E-02	0.0016868402	6.92517714393E-05 → 6.92517714393E-05
1	0.0013809723	0.0013271505	3.8973809289E-02	0.0013810679	6.92517714430E-05 → 6.92517714430E-05

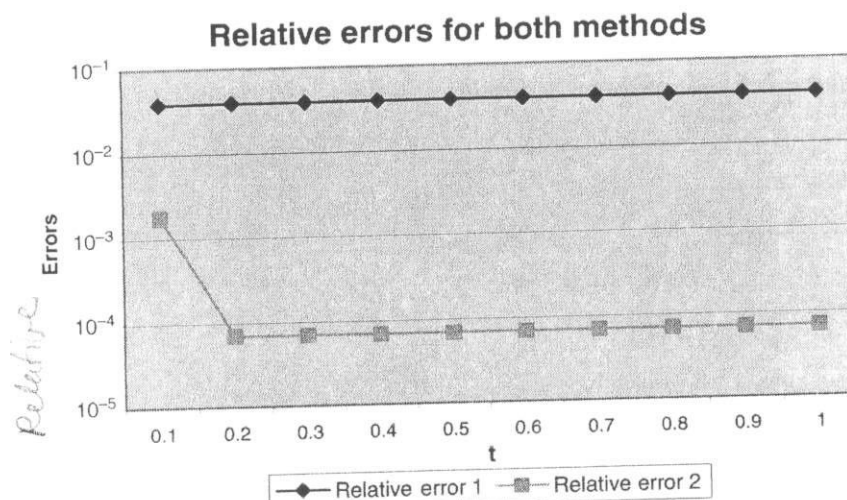


Figure 3. Relative errors for the third order composite Arithmetic-Harmonic Runge-Kutta and the modified third order mean Runge-Kutta (GM<sup>2</sup>/AM) methods.

## 5. Conclusion

We have shown that it is possible to construct a composite method for solving stiff problems from the various forms of third-order means of the Runge–Kutta formulation. The combination of arithmetic and harmonic means provides a new explicit method for the solution of stiff ordinary differential equations which could reduce the cost of computation compared with implicit methods.

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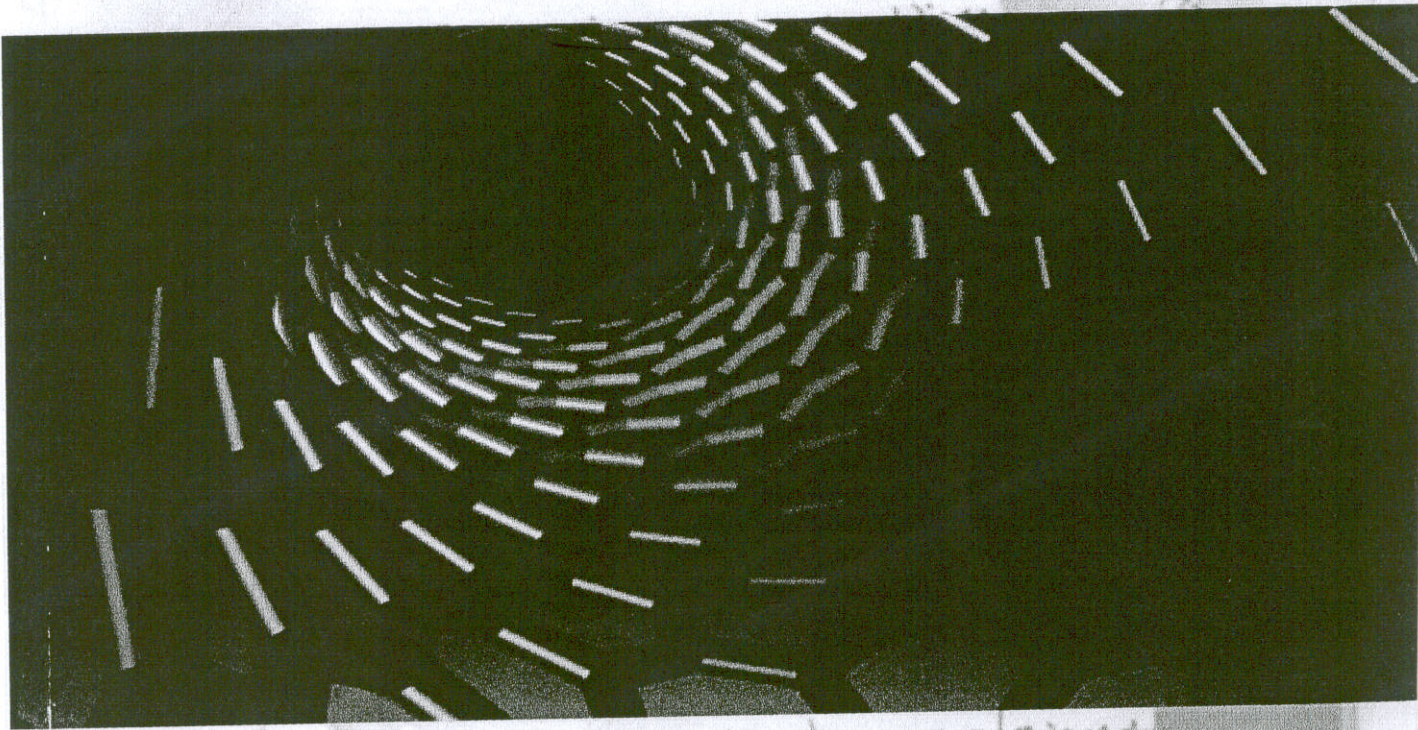
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## Sin-Cos-Taylor-Like method for solving stiff ordinary differential equations

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### ABSTRACT

This paper discusses the derivation of an explicit Sin-Cos-Taylor-Like method for solving stiff ordinary differential equations, which is a formulation of the combination of a polynomial and the exponential function. This new method requires extra work to evaluate a number of differentiations of the function involved. However, the result shows smaller errors when compared to the results from the explicit classical fourth-order Runge-Kutta (RK4) and the Adam-Bashforth-Moulton (ABM) methods. Implicit methods could work well for stiff problems but have certain drawbacks especially when discussing about the cost. Although extra work is required, this explicit method has its own advantages. Besides providing excellent results, the cost of computation using this explicit method is much cheaper than the implicit methods. We also considered the stability property for this method since the stability property of the classical explicit fourth order Runge-Kutta method is not adequate for the solution of stiff problems. As a result, we find that this explicit method is of order  $\infty$ , which has been developed, and proved to be both A-stable and L-stable.

5

| Stiff ordinary differential equations | Explicit methods | A-stable | L-stable |

### 1. Introduction

Stiff problem entails rapidly decaying transient solution, which arises naturally in wide variety of applications including the study of spring and damping system, the analysis of control system and problems in the chemical kinetics [1]. Stiff differential equations also occur in other kind of studies, such as biochemistry, biomedical system, weather prediction, mathematical biology and electronics. In chemical kinetics, stiffness is caused in the vast majority of cases merely by a great difference among the reaction rate constants. This problem is more likely to occur whenever we have a larger system or the more detail and complicated the models are. The atmospheric phenomena as an example, involves transport with chemical reaction, thus stiffness can occur because of the time scales of the reactions are much smaller than times for movement over distances. Stiffness in heat transfer originates physically in one of two ways; sharp changes in the thermal environment of large differences in the rates which components of the system can transfer heat [2].

Stiffness is generally understood in terms of what goes wrong when numerical methods not design for such problems are used to try to solve them [3]. Lambert in [4] points out that one should consider stiffness as a phenomenon exhibited by a system, rather than a property of it, because the word property is associated to the



existence of a definition, which is both comprehensive and precise, whereas it is difficult to come out with a satisfactory definition for the concept of "stiffness".

This paper focuses on the explicit one-step methods for solving stiff differential equations. Although most of the numerical analysts were confident that the implicit methods work better in producing results for stiff problems but this research meant to explore the explicit methods, which recently was proven, could also satisfy the stiff problems [5].

## 2. Formulation

Rokiah & Nazeeruddin [6] have shown that the explicit one-step method for stiff problems could be represented by the composition of a polynomial and exponential function of the form

$$PE(t) = a_0 + t(a_1 + t(a_2 + t(a_3 + t(a_4 + a_5 t)))) + Ab_1 e^{b_2 t} \quad (1.1)$$

Taking  $A = 1$ , we calculated the values of  $a_i$ ,  $i = 0, 1, 2, 3, 4, 5$  and  $b_i$ ,  $i = 1, 2$ . In [7], we substituted the constant  $A = 1$  with a trigonometric function,  $\sin z_n h$ . Based on the same theory for the solution of a differential equation with complex eigenvalues, we replaced the constant  $A$  by  $\sin z_n h + \cos z_n h$  which produced a Sin-Cos-Taylor-Like method. For simplicity we name this method as SCTL6 method.

Provided that  $f^{(5)}, f^{(6)} \neq 0$ , we obtained the equation

$$PE(t) = y_n + (t - t_n) \left( f_n + (t - t_n) \left( \frac{f'_n}{2} + (t - t_n) \left( \frac{f''_n}{6} + (t - t_n) \left( \frac{f'''_n}{24} + (t - t_n) \frac{f^{(4)}_n}{120} \right) \right) \right) \right) + \frac{f^{(5)}_n (\sin(z_n h) + \cos(z_n h))}{z_n^6} \left( e^{z_n(t-t_n)} - 1 - z_n(t-t_n) - \frac{1}{2}(z_n(t-t_n))^2 - \frac{1}{6}(z_n(t-t_n))^3 - \frac{1}{24}(z_n(t-t_n))^4 - \frac{1}{120}(z_n(t-t_n))^5 \right) \quad (1.2)$$

where

$$z_n = \frac{f^{(6)}_n}{f^{(5)}_n}$$

Letting  $t = t_{n+1}$ , we arrive at the formula below:

$$y_{n+1} = y_n + h \left( f_n + h \left( \frac{f'_n}{2} + h \left( \frac{f''_n}{6} + h \left( \frac{f'''_n}{24} + h \frac{f^{(4)}_n}{120} \right) \right) \right) \right) + \frac{f^{(5)}_n (\sin(z_n h) + \cos(z_n h))}{z_n^6} \left( \exp(z_n h) - 1 - h z_n \left( \frac{1}{2} + h z_n \left( \frac{1}{6} + h z_n \left( \frac{1}{24} + \frac{z_n h}{120} \right) \right) \right) \right) \quad (1.3)$$

### 3. Local Truncation Error

The local truncation error for this particular method given by the formula (1.3) can be represented by

$$\begin{aligned}
 T_{n+1} &= y(t_{n+1}) - y_{n+1} \\
 &= y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(t_n) + \frac{h^4}{24} y^{(4)}(t_n) + \frac{h^5}{120} y^{(5)}(t_n) + \frac{h^6}{720} y^{(6)}(t_n) + \\
 &\quad O(h^7) - y_n - h \left( y'_n + h \left( \frac{y''_n}{2} + h \left( \frac{y'''_n}{6} + h \left( \frac{y^{(4)}_n}{24} + h \frac{y^{(5)}_n}{120} \right) \right) \right) \right) - \\
 &\quad \frac{y_n^{(6)} (\sin(z_n h) + \cos(z_n h))}{z_n^6} \left( e^{z_n h} - 1 - z_n h \left( 1 + z_n h \left( \frac{1}{2} + z_n h \left( \frac{1}{6} + z_n h \left( \frac{1}{24} + \frac{z_n h}{120} \right) \right) \right) \right) \right) \\
 &= \frac{h^6}{720} y_n^{(6)} (1 - (\sin(z_n h) + \cos(z_n h))) + O(h^7).
 \end{aligned}$$

With  $T_{n+1} = \frac{h^6}{720} y_n^{(6)} (1 - (\sin(z_n h) + \cos(z_n h))) + O(h^7)$ , we conclude that the SCT-L6 method is a <sup>fifth</sup> ~~sixth~~-order method.

### 4. Stability

#### 4.1. Theorem 1.1

The explicit Sin-Cos-Taylor-Like method is  $A$ -stable.

Proof:

Applying equation (0.3) to the test equation, with  $\text{Re}(\lambda) < 0$ , we obtain

$$\begin{aligned}
 y_{n+1} &= y_n + h \left( \lambda y_n + h \left( \frac{\lambda^2 y_n}{2} + h \left( \frac{\lambda^3 y_n}{6} + h \left( \frac{\lambda^4 y_n}{24} + h \frac{\lambda^5 y_n}{120} \right) \right) \right) \right) + \\
 &\quad y_n (\sin(z_n h) + \cos(\lambda h)) \left( e^{\lambda h} - 1 - \lambda h \left( 1 + \lambda h \left( \frac{1}{2} + \lambda h \left( \frac{1}{6} + \lambda h \left( \frac{1}{24} + \frac{\lambda h}{120} \right) \right) \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
&= y_n \left( 1 + h\lambda + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{120} \right) + \\
&\quad y_n (\sin(\lambda h) + \cos(\lambda h)) \left( e^{\lambda h} - \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{120} \right) \right) \\
&= y_n \left( e^{\lambda h} (\sin(\lambda h) + \cos(\lambda h)) + (1 - (\sin(\lambda h) + \cos(\lambda h))) \right) \\
&\quad \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!} \right)
\end{aligned}$$

which leads us to

$$y_{n+1} = G(\lambda h) y_n;$$

$$\begin{aligned}
G(\lambda h) &\approx e^{\lambda h} [\sin(\lambda h) + \cos(\lambda h)] + [1 - (\sin(\lambda h) + \cos(\lambda h))] e^{\lambda h} \\
&= e^{\lambda h}
\end{aligned}$$

$$\text{Since } y_{n+1} \approx e^{\lambda h} y_n$$

$$y_1 \approx e^{\lambda h} y_0; \quad y_2 \approx e^{\lambda h} y_1 \approx e^{2\lambda h} y_0 \quad \dots \quad y_k \approx e^{\lambda h} y_{k-1} \approx e^{k\lambda h} y_0$$

For any fixed point  $t = t_n = nh$ , we have

$$y_n \approx e^{n\lambda h} y_0.$$

Since  $|e^{n\lambda h}| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\lambda h$  with  $\text{Re}(\lambda) < 0$ , we have  $y_n \rightarrow 0$  as  $n \rightarrow \infty$  and consequently the method is  $A$ -stable.

#### 4.2. Theorem 1.2

The explicit Sin-Cos-Taylor-Like method is also  $L$ -stable.

Proof:

Applying equation (0.3) to the test equation, with  $\text{Re}(\lambda) < 0$ , we obtain

$$y_{n+1} \approx e^{\lambda h} y_n$$

From Theorem 1.1, the method is  $A$ -stable.

Since  $|e^{\lambda h}| \rightarrow 0$  as  $\text{Re}(\lambda h) \rightarrow -\infty$ , we have  $L$ -stability.

We plot the stability region for the SCLT6 method as given in Figure 1 and 2. Using MATHEMATICA [8].

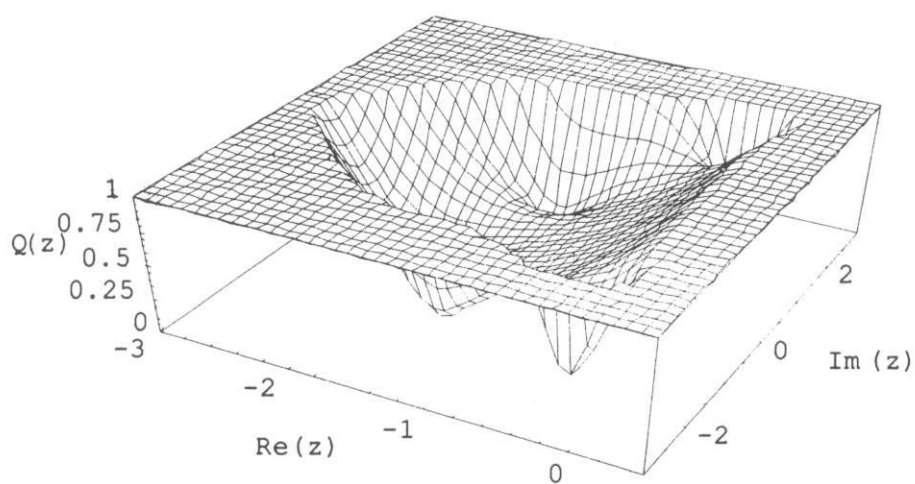


Fig. 1 The stability region of the SCTL6 method in 3D.

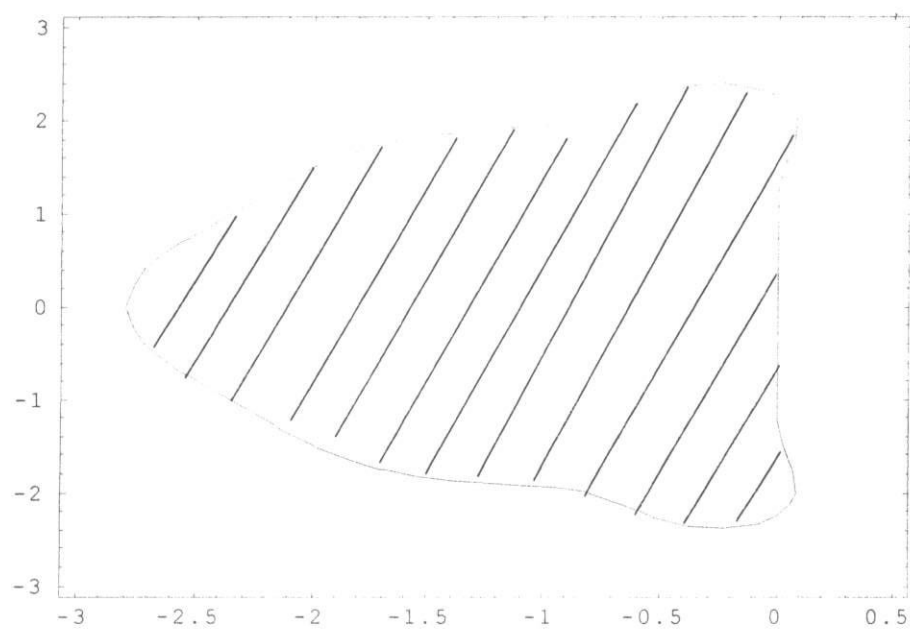


Fig. 2 The stability region in 2D given by the SCTL6 method.

## 5. Numerical Results and Discussion

The formula [1.4] is tested on the stiff ordinary differential equation

$$y'(t) = -100y(t) + 99e^{-t}, \quad y(0) = 0, \quad (3.1)$$

and compared to the exact solution using the step size,  $h = 0.01$  and  $h = 0.02$ . We also solve equation (3.1) using another two established methods, namely the classical fourth-order Runge-Kutta (RK4), the implicit Adam-Bashforth-Moulton (ABM) methods. The relative errors for the methods applied, with the two different step sizes are compared and presented in Table 1 and 2. Table 3 shows the number of function evaluation in each iteration for the RK4, ABM and SCTL6 methods.

Figure 3 and 4 exhibits the graphs of the relative errors for all the methods used while in Figure 5 and 6, we illustrate the graphs of the exact solution together with the RK4, ABM and SCTL6 methods using different value of step size, i.e.  $h = 0.01$  and  $h = 0.02$ .

**Table 1** Explicit Sin-Cos-Taylor-Like method,  $h = 0.01$  on the equations  $y' = -100y(t) + 99\exp(-t)$  compared to the classical Runge-Kutta (RK4) and Adam Bashforth-Moulton (ABM) methods.

t	Exact solutions	Relative Error ( $h = 0.01$ )		
		RK4	ABM	CTL6
0.1	0.90479201811	3.2843157707E-03	3.3420582518E-05	1.41700453180E-05
0.2	0.81873075102	5.3589898295E-04	4.6435765374E-06	1.66146927457E-09
0.3	0.74081822068	6.8201816855E-06	4.6488512241E-06	2.13272074342E-12
0.4	0.67032004604	1.0804073360E-05	4.6488430870E-06	6.49236623601E-12
0.5	0.60653065971	1.1613090101E-06	4.6488539332E-06	4.35427045393E-12
0.6	0.54881163609	6.2829963309E-08	4.6488569286E-06	7.34859812373E-12
0.7	0.49658530379	2.9160200451E-08	4.6488524296E-06	2.85154221430E-12
0.8	0.44932896412	2.2715739765E-09	4.6488434082E-06	6.17143400501E-12
0.9	0.40656965974	2.4400746564E-10	4.6488510658E-06	1.48564169037E-12
1.0	0.36787944117	9.6759880091E-11	4.6488535118E-06	3.93337875730E-12

**Table 2** Explicit Sin-Cos-Taylor-Like method,  $h = 0.02$  on the equations  $y' = -100y(t) + 99\exp(-t)$  compared to the classical Runge-Kutta (RK4) and Adam Bashforth-Moulton (ABM) methods.

t	Exact value	Relative Error (h = 0.02)		
		RK4	ABM	SCTL6
0.1	0.90479201811	4.4471205718E-03	3.1145021711E-01	5.0186660602E-05
0.2	0.81873075102	3.0506054044E-05	9.9141115426E-01	2.5135869371E-09
0.3	0.74081822068	5.1095079591E-05	2.5565149730E+00	2.3163046736E-12
0.4	0.67032004604	5.1188730382E-05	3.2739007255E+00	6.5087631915E-12
0.5	0.60653065971	5.1189156315E-05	1.0065478912E+02	4.3394438226E-12
0.6	0.54881163609	9.3271280234E-02	8.5027262232E+02	7.3338305285E-12
0.7	0.49658530379	5.1189158233E-05	5.1817251752E+03	2.8353332833E-12
0.8	0.44932896412	5.1189158258E-05	2.5086071824E+04	6.1870003397E-12
0.9	0.40656965974	5.1189158252E-05	9.2247557198E+04	1.4708958671E-12
1.0	0.36787944117	5.1189158261E-05	1.7389140715E+05	3.9170821023E-12

**Table 3** Number of functions evaluation in the methods used.

Methods	Function evaluation in one iteration
Classical Runge-Kutta	5
Adams-Bashforth-Moulton	6
Sin-Cos-Taylor-Like	6

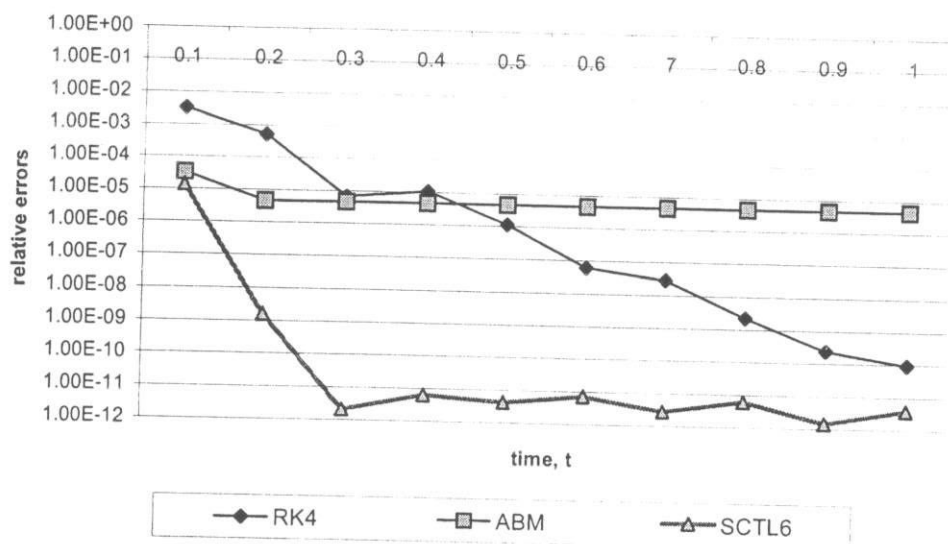


Fig. 3 Relative errors for the four methods used to solve the stiff problem using  $h = 0.01$ .

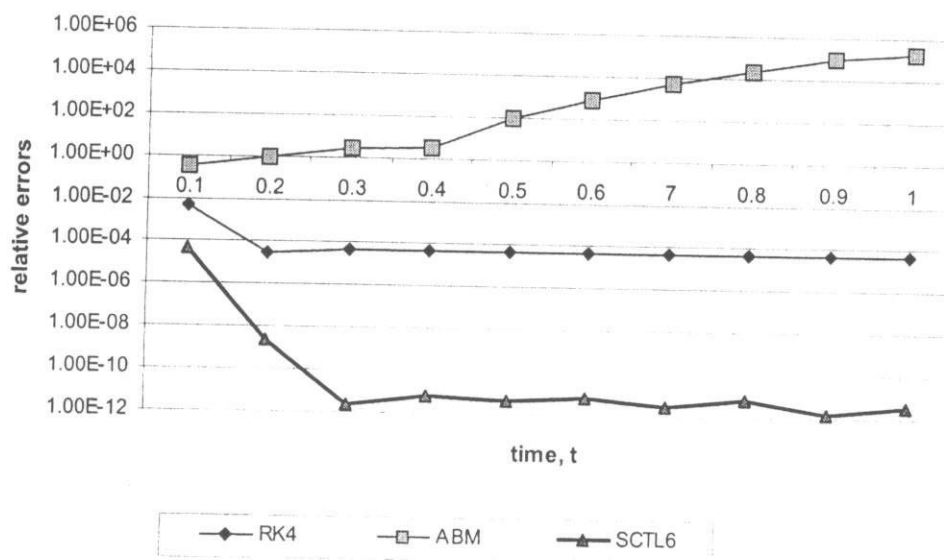


Fig. 4 Relative errors for the four methods used to solve the stiff problem using  $h = 0.02$ .

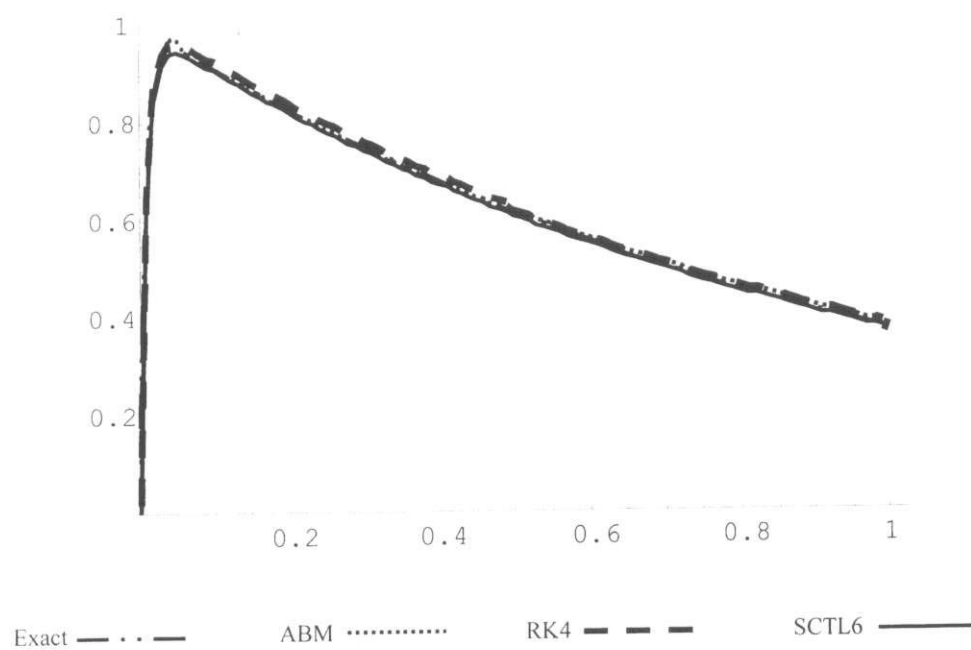


Fig. 5 Graphs of the exact value together with the four methods used to solve the stiff problem for  $h = 0.01$ .

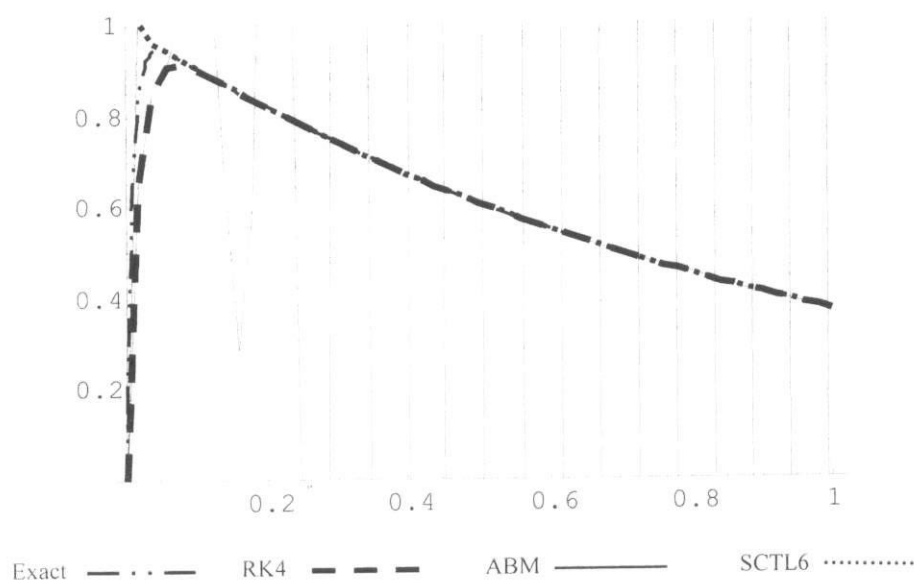


Fig. 6 Graphs of the exact value together with the four methods used to solve the stiff problem for  $h = 0.02$ .



## 6. Conclusion

This research generally discusses a one-step explicit method involves in solving stiff ordinary differential equations, namely, the Sin-Cos-Taylor-Like (SCTL6) method. The results show excellent accuracy of the SCTL6 method using two step sizes,  $h = 0.01$  and  $h = 0.02$ , compared to the classical RK4 and the ABM methods. The difference of the step size in SCTL6 does not show any significant difference of the error obtained. Nevertheless, RK4 method shows a better results when  $h = 0.01$  compared to  $h = 0.02$ , which shows that this method requires more iteration to obtain a better accuracy. The stability region for the fourth order RK4 and the ABM methods are given by  $|\lambda h| < 2.78$  and  $|\lambda h| < 1.25$  respectively [9]. In the stiff differential equation above, with  $\lambda = -100$ , the range for the RK4 step size is  $h < 0.0278$  whereas the range for the ABM method is  $h < 0.0125$ . Since  $h = 0.02$  is greater than  $0.0125$  and vice versa for  $h = 0.01$ , we find that the ABM method could only produce better results when the step size is  $0.01$ . It is also proven that SCTL6 method has the  $A$ -stable and  $L$ -stable properties.

The results obtained are worth the work done on the differentiations. We realize that the function evaluations of each method is more or less the same but the cost of computation, which includes the number of iteration needed is much 'cheaper' in the explicit method compared to the implicit ones. We conclude that the RK4 and ABM could only reach to a comparable accuracy with the SCTL6 method when the step size,  $h$  is small enough and that makes the latter better than the RK4 and ABM methods.

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