STUDY ON SOLUTIONS OF HEAT PROBLEMS USING FINITE DIFFERENCE METHODS AND METHOD OF LINES INCORPORATE WITH RK-LIKED METHODS

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To my beloved family and friends....

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ABSTRACT

This dissertation reports a comparison of results from two classes of numerical methods for heat problems. The heat or diffusion equation, which is an example of parabolic equations are classified into the categories of the partial differential equations. Two classes of numerical methods, Method of Lines and Finite Difference Method are discussed. In Method of Lines, several Runge-Kutta methods were incorporated, including the third and fourth order. Finally, analysis on numerical results for the three heat problems is presented.

ABSTRAK

Disertasi ini melaporkan dapatan kajian daripada dua kaedah berangka bagi menyelesaikan masalah haba. Masalah haba merupakan contoh persamaan parabolik dan dikategorikan dalam persamaan terbitan separa. Dua kaedah berangka iaitu Kaedah Garis dan Kaedah Pembeza Terhingga dibincangkan. Dalam Kaedah Garis, beberapa Kaedah Runge-Kutta digunakan termasuk Kaedah Runge-Kutta peringkat ketiga dan keempat. Akhir sekali, analisis ke atas dapatan berangka untuk tiga masalah haba dihuraikan.

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CHAPTER I

RESEARCH FRAMEWORK

1.1 Introduction

The study of heat problems started since in the 18^{th} century. A large number of papers have been published on numerical methods for heat problems. However, papers comparing these methods are usually restricted to an examination of the stability of the finite difference schemes used by the methods [Price, 1974].

This study reports a comparison of solutions from two classes of the more important numerical methods for heat problem which is The Method of Lines and Finite Difference Method. We will compare the accuracy, computational complexity, stability and simplicity of the methods for the solutions of heat problems.

Forward Finite Difference Method (FFDM) and Centered Finite Difference Method (CFDM), which are the two approximations in the Finite Difference Method will be considered. Using The Method of Lines, will we considered the Runge-Kutta methods to find the solutions, which are Heun's Third-Order Formula (HTOF), Kutta's Third Order Rule (KTOR), Stage-Arithmetic Mean Runge-Kutta Method (SAMRKM) and Classical Fourth-Order Runge-Kutta Method (CFORKM). We have considered the Runge-Kutta methods because the methods are the most popular choice for numerical solution of Ordinary Differential Equations (ODE) in the context of Method of Lines.

1.2 Problem Statement

The problem can be stated as follows :

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad k > 0. \tag{1.1}$$

The partial differential equation (1.1) is known as the one-dimensional heat equation. Equation (1.1) occurs in the theory of heat flow – that is, heat transferred by conduction in a rod or in a thin wire. The function $u(x,t)$ represents temperature at a point x along the rod at some time t.

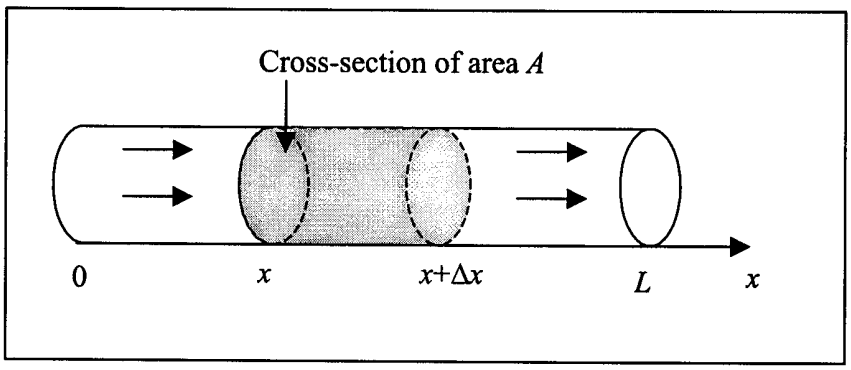

Figure 1.1 : A thin circular rod

We have to make many simplifying assumptions; it is worthwhile to see how equation (1.1) arises. Suppose a thin circular rod of length L has a cross-sectional area A and coincide with the x-axis on the interval [0, L]. See Figure 1.1. Let us suppose [Zill and Cullen, 2001] :

- The flow of heat within the rod takes place only in the x-direction.
- The lateral, or curved, surface of the rod is insulated; that is, no heat escapes from this surface.
- No heat is being generated within the rod.
- The rod is homogeneous: that is, its mass per unit volume ρ is a constant.

- The specific heat γ and thermal conductivity K of the material of the rod are constant.

To derive the partial differential equation satisfied by the temperature $u(x,t)$ we need two empirical laws of heat conduction [Zill and Cullen, 2001] :

i. The quantity of heat Q in an element of mass m is

$$Q = \gamma m u, \tag{1.2}$$

where u is the temperature of the element.

ii. The Rate of heat flow Q_t through the cross-section indicated in Figure 1.1 is proportional to the area A of the cross-section and the partial derivative with respect to x of the temperature :

$$Q_t = -KAu_x . \tag{1.3}$$

Since heat flows in the direction of decreasing temperature, the minus sign in (1.3) is used to ensure that Q_t is positive for $u_x < 0$ (heat flow to the right) and negative for $u_x > 0$ (heat flow to the left). If the circular slice of the rod shown in Figure 1.1 between x and $x + \Delta x$ is very thin, then $u(x,t)$ can be taken as the approximate temperature at each point in the interval. Now the mass of the slices is $m = \rho(A\Delta u)$, and so it follows from (1.2) that the quantity of heat in it is

$$Q = \gamma \rho A \Delta x u . \tag{1.4}$$

Furthermore, when heat flows in the positive x-direction, we see from (1.3) that heat builds up in the slice at the net rate

$$-KAu_x(x,t) - [-KAu_x(x+\Delta x,t)] = KA[u_x(x+\Delta x,t) - u_x(x,t)]. \tag{1.5}$$

By differentiating (1.6) with respect to t we see that the net rate is also given by

$$Q_t = \gamma\rho A\Delta x u_t. \tag{1.6}$$

Equating (1.5) and (1.6) gives

$$\frac{K}{\gamma\rho}\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x} = u_t. \tag{1.7}$$

Taking the limit of (1.7) as $\Delta x \to 0$ finally yields (1.1) in the form

$$\frac{K}{\gamma\rho}u_{xx} = u_t. \tag{1.8}$$

It is customary to let $k = \frac{K}{\gamma\rho}$ and call this positive constant the thermal diffusivity [Zill and Cullen, 2001].

In this study, we shall be concerned with finding solutions of the three problems of the one dimensional heat equation with two classes of numerical methods, namely Method of Lines and Finite Difference Method. Our work involves the comparison of these solutions. The three problems are :

1. Problem I

The heat problem :

$$u_t = ku_{xx}; \qquad 0 < x < 1$$

with

$$\begin{aligned} &u(0,t) = 50^0 C, \\ &u(1,t) = 20^0 C; \quad 0 < t < 0.3 \\ &u(x,0) = 70^0 C; \quad 0 \le x \le 1 \end{aligned} \tag{1.9}$$

and $k = 0.1$ [Rokiah *et al*, 2002]. The problem is considered because the initial value condition consist a constant value.

2. Problem II

The boundary value problem :

$$u_t = ku_{xx}; \qquad 0 < x < 1$$

with

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= 0; \qquad 0 < t < 0.4 \\ u(x,0) &= \sin(\pi x); \qquad 0 \le x \le 1 \end{aligned} \tag{1.10}$$

and $k = 1$ [Zill and Cullen, 2001]. The problem is considered because the initial value condition consist an oscillation function.

3. Problem III

The heat problem :

$$u_t = ku_{xx}; \qquad 0 < x < 5$$

with

$$\begin{aligned} u(0,t) &= 10, \\ u(5,t) &= 30; \qquad 0 < t < 0.30 \\ u(x,0) &= 10 - 2x; \qquad 0 \le x \le 5 \end{aligned} \tag{1.11}$$

and $k = 1.0$ [McOwen, 1996]. The problem is considered because the initial value condition consist a linear function.

Some problem statements will be considered. The study is focus on the questions as follows :

1. Which is the best numerical method to solve the heat problem between The Method of Lines and Finite Difference Method ?

2. How the procedures of The Method of Lines and Finite Difference Method to solve the heat problem ?
3. What the advantages and disadvantages of the methods ?
4. How to analysis the accuracy, computational complexity, simplicity and stability of the methods ?

1.3 Analysis Factors

The aim of the investigations presented was to study the solutions of the two classes of numerical methods, namely Method of Lines incorporate with RK-liked methods and The Finite Difference Methods with respect to their ability to simulate the heat problems.

The following points should be looked at in order to judge the applicability of a numerical method:

1. Accuracy of the solution
2. Computational complexity
3. Stability
4. Simplicity of the methods

1.3.1 Accuracy of the Solution

A common way of classifying and comparing methods is to give their order of accuracy [Smith, 1979]. For our purpose, the order of a method qualitatively describes the errors expression. The errors analysis of computations is characterized according to several different criteria. Let x denote an exact value and let variable x^* denote an approximation of x. One may think of x^* as the result of a computation. The errors which are consider in this study as follows :

1. Average mean square error (*amse*),

   The *amse* is defined to be

$$amse = \frac{\sum_{i=0}^{N} |x - x^*|}{N}$$

   where N is a number of the solutions. In the comparison of the numerical methods, the lowest value of *amse* is the best method.

2. Maximum absolute error (*mae*),

   The absolute error (*ae*) is defined to be

$$ae = |x - x^*|,$$

   and the maximum absolute error is a maximum or the biggest value of the solutions for a numerical method [Zill and Cullen, 2001].

3. Root mean square error (*rmse*),

   The *rmse* is defined to be

$$rmse = \left(\frac{\sum_{i=0}^{N} (x - x^*)}{N} \right)^{1/2}$$

   where N is a number of the solutions.

4. Maximum relative error (*mre*) .

   The relative error (*re*) is defined to be

$$re = \frac{ae}{|x|} = \frac{|x - x^*|}{|x|}$$

   where the maximum relative error is a maximum or the biggest value of the solutions for a numerical method [Zill and Cullen, 2001]. The relative error is usually the more important error measure : one wants to know how many digits in a computed result are correct [Yakowitz and Szidarovszky, 1986].

1.3.2 Computational Complexity

Computational complexity is a total of arithmetic operation numbers for any two real numbers [Norma, 2004]. The arithmetic operation such as plus (+), minus (-), multiply (x) and divide (÷). Application of trigonometry functions such as sinus, cosinus and tangent, exponent function and absolute function also effect to the computational complexity.

1.3.3 Stability.

An important consideration in using numerical methods to approximate the solution of an initial-value problem is the stability of the method. Simply stated, a numerical method is stable if small changes in the initial condition result in only small changes in computed solution [Zill and Cullen, 2001]. A numerical method is said to be unstable if it is not stable.

A numerical method is unstable if round-off errors or any other errors grow too rapidly as the computations proceed [Zill and Cullen, 2001].

1.3.4 Simplicity of the Method.

Simplicity of the method is a degree of hardness in implements the methods. We can conclude about the simplicity of the method by considered the computational complexity and the time consuming.

1.4 Objective

The objectives of this study are as follows:

1. To study the partial differential equations and their examples.
2. To study the procedure of the Method of Lines and the Runge-Kutta Methods.
3. To study the procedure of the Finite Difference Methods.
4. To find the better numerical method.
5. To compare the advantages and disadvantages between the methods.

1.5 Scope

Our work is focusing in the comparison of the solution from the numerical experiments of the heat problems. The comparison is between the Finite Difference Methods and Methods of Lines incorporate with RK-Liked Methods.

1.6 Thesis Organization

Chapter II starts with the introduction of partial differential equations. This is followed by some examples of the partial differential equations. The procedure to find the exact value of partial differential equations using the separation of variables is discussed.

Chapter III begins with an introduction to the Methods of Lines and an example of the heat equation to illustrate the procedure of the Method of Lines in partial differential equations. It then touches briefly on the Runge-Kutta (RK) methods and the Runge-Kutta Methods which used in the numerical experiment for the heat problems.

Chapter IV is concerned with the Finite Difference Methods. It contains some fundamental in the Finite Difference Methods. Then, we describe the procedure of Explicit Methods and Implicit Methods. This is followed by the discussion of convergence and stability of the Finite Difference Methods.

In chapter V, we discuss briefly the numerical experiments for the three heat problems using Method of Lines and Finite Difference Methods. Method of Lines will incorporate with Heun's Third-Order Formula, Kutta's Third-Order Rule, Stage-Arithmetic Mean Runge-Kutta Method and Classical Fourth Order Runge-Kutta Method. The Finite Difference Methods will use the Forward Difference Formula and Centered Difference Formula.

Chapter VI discusses the comparison of the solution from the numerical experiment in Chapter V and summarizes the conclusions from the comparison. Finally, we summarized the work and give the conclusions. The future works were mentioned here. This is followed by references and appendices.

7. The Final conclusion is The Finite Difference Method is the best method for desk computation but The Method of Lines is the best method for computerize.

6.3 Suggestions

1. Application of 5-points finite difference approximation for the Method of Lines may produce more accurate result. This is because we can reduce the errors by using 5-points finite difference approximation. But the method will become more difficult and we need to use software.

2. Using parallel algorithm for Method of Lines with large size of matrix. For example, the 5-points finite difference approximation for the Method of Lines will produce a large size of matrix. We can produce a parallel algorithm using the Parallel Virtual Machine (PVM).

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