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# TWO-SOLITON SOLUTIONS OF THE KADOMTSEV-PETVIASHVILI EQUATION

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**Abstract.** Several findings on soliton solutions generated by the Kadomtsev-Petviashvili (KP) equation were discussed in this paper. This equation is a two dimensional of the Korteweg-de Vries (KdV) equation. Traditional group-theoretical approach can generate analytic solution of solitons because KP equation has infinitely many conservation laws. By using Hirota Bilinear method, we show via computer simulation how two solitons solution of KP equation produces triad, quadruplet and a non-resonance structures in soliton interactions.

*Keywords:* Soliton, Hirota Bilinear method, Korteweg-de Vries and Kadomtsev-Petviashvili equations

**Abstrak.** Beberapa keputusan tentang penjanaan penyelesaian soliton oleh persamaan Kadomtsev-Petviashvili akan dibincangkan dalam kertas ini. Kaedah teori kumpulan mampu memberikan penyelesaian secara analitik kerana persamaan KP mempunyai ketakterhinggaan banyaknya hukum keabadian. Dengan kaedah Bilinear Hirota, ditunjukkan melalui simulasi berkomputer bagaimana penyelesaian dua soliton persamaan KP mampu menghasilkan struktur-struktur "triad", kuadruplet dan struktur tak beresonan dalam interaksi soliton.

*Kata kunci:* Soliton, kaedah Bilinear Hirota, persamaan Kortewegde Vries dan Kadomtsev-Petviashvili

## 1.0 INTRODUCTION

There are many examples of resonance in physics. However, resonance in soliton interaction is an interesting phenomena. In this paper, we will use the Kadomtsev-Petviashvili (KP) equation to model the two-soliton interactions as in Ong [1] and also in Anker and Freeman [2]. In particular, the KP equation is the two-dimensional form of the Korteweg-de Vries (KdV) equation. Miles [3, 4] discovered that in the interaction of two solitons, the interaction region between the incident solitons and the centered-shifted solitons after interaction is essentially itself a single soliton which leads to a very simple conceptual picture of the interaction process. This interacting soliton is the resonant soliton associated with the resulting solitary waves form by two incident solitons.

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## 2.0 TWO-SOLITON SOLUTIONS- KP EQUATION

Generally, the KP equation can be written as:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial y^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

and can be simply written as:

$$\left(u_t+6\,u\,u_x+u_{xxx}\right)_x+3\,u_{yy}=0 \tag{2}$$

We will consider the linearized form of Equation (2), then we have plane-wave solutions whose phase variable $kx + my - wt$ satisfied the dispersion relation

$$w=\frac{3m^2}{k}-k^3 \tag{3}$$

$N$ – soliton solutions for Equation (2) had been solved by Satsuma [5] using Hirota Bilinear method [6] as:

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln f \tag{4}$$

$$=2\left[\frac{f_{xx}f-f_x^2}{f^2}\right] \tag{5}$$

with the function $f$ given by

$$f=\left[\delta_{ij}+\frac{a_i}{l_i+n_j}\exp(\eta_i)\right] \tag{6}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j,\\ 0 & i\neq j,\end{cases}$$

and

$$\eta_i=k_i x+m_i y-w_i t \tag{7}$$

with

$$k_i=l_i+n_i, \tag{8}$$

$$m_i=n_i^2-l_i^2 \tag{9}$$

$$w_i=\frac{3m_i^2}{k_i}-k_i^3 \tag{10}$$

and

$$\varepsilon_i = \frac{a_i}{l_i + n_i} \quad \text{for} \quad i, j = 1 \text{ and } 2. \tag{11}$$

Thus,

$$f = 1 + \varepsilon_1 \exp(\eta_1) + \varepsilon_2 \exp(\eta_2) + A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2) \tag{12}$$

where

$$A_{12} = \frac{(n_1 - n_2)(l_1 - l_2)}{(l_1 + n_2)(l_2 + n_1)} \tag{13}$$

By using Microsoft Visual C++ Professional Edition, we will generate a computer programming to portray the 2-soliton solutions of Equation (2) and later produce various Encapsulated Postscript (eps) files. These results were demonstrated in computer simulation section.

## 3.0 RESONANCES IN THE KP EQUATION

There are three types of resonances in KP solitons interaction which are full resonance, partial resonance and non-resonance as in Ong [1]. Resonance will only occurs when the value of $A_{12}$ is very close to zero. Therefore, the values of $n_1$, $n_2$, $l_1$ and $l_2$ will determine the resonance structure. If we fix the values of $l_1$ and $l_2$, then $n_1$ and $n_2$ will determine the value of $A_{12}$. We will show this effect in the following section.

### 3.1 Full Resonance

We need to have a situation where $n_1 = n_2$ or $l_1 = l_{21}$ so that ln $A_{12}$ tends to $-\infty$ or simply $A_{12}$ will be zero. Therefore, we shall fix the values of $l_1$ and $l_2$ but $l_1 \neq l_2$ and consider the case where $n_1 = n_2$. Therefore, Equation (12) will become:

$$f = \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)} \tag{14}$$

Any combinations between (1), (2) and (3) will form a new soliton. We can observe that the first soliton, $S_1$, second soliton, $S_2$ and the resonant soliton, $S_{23}$ can be represented by the function f as below.

$$\text{Soliton 12}: \; f = 1 + \varepsilon_1 \exp(\eta_1) \tag{15}$$

$$\text{Soliton 13}: \; f = 1 + \varepsilon_2 \exp(\eta_2) \tag{16}$$

$$\text{Soliton 23}: \; f = \varepsilon_1 \exp(\eta_1) + \varepsilon_2 \exp(\eta_2) \tag{17}$$

When the term $\ln A_{12}$ tends to $-\infty$, the length of resonant soliton will increase and this will form a triad as shown in Figure 1.

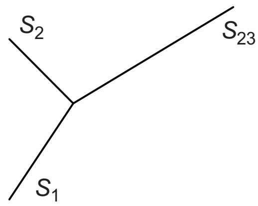



**Figure 1** A triad

### 3.2 Partial Resonance

In this case, we will consider the case of $n_1 \approx n_2$ so that the values of $A_{12}$ will be so close to zero. Thus we have Equation (12) again with $A_{12} \approx 0$. The value of $A_{12}$ will determine the length of resonant soliton $S_{23}$. In this case, we will have an interaction pattern called quadruplet as shown in Figure 2, which represents the partial resonant structure.

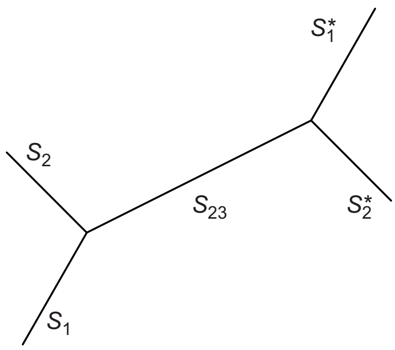



**Figure 2** A quadruplet

From Figure 2, when $S_1$ and $S_2$ come into interaction, it will produce a resonant soliton $S_{23}$ and later break up again to form $S_1^*$ and $S_2^*$which are actually soliton $S_1$ and $S_2$ respectively, but of course with some phase shift.

### 3.3 Non-Resonance

For the non-resonance case, we will take $n_1 \neq n_2$ that $A_{12} \approx 1$. This means that the line $S_{23}$ no longer exists. This phenomena is called non-resonance interactions. In this cases, $S_1$ will be centered along the line $k_1x + m_1y = 0$ whereas $S_2$ will be centered along the line $k_2x + m_2y = 0$ with

$$\tan\alpha = \frac{\beta_1 - \beta_2}{1 + \beta_1\beta_2}, \qquad \text{where} \quad \beta_1 = \frac{m_1}{k_1}, \quad \beta_2 = \frac{m_2}{k_2}. \tag{18}$$

In this case, $\alpha$ is the angle of interaction between $S_1$ and $S_2$ and $\beta_1$, $\beta_2$ are the tangents of the lines respectively. In this case, we will have an interaction pattern called cross as shown in Figure 3, which represents the non-resonance structure.

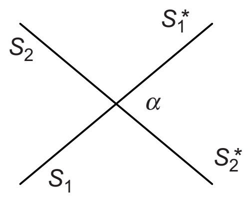



**Figure 3** A cross

## 4.0 COMPUTER SIMULATIONS

By using Microsoft Visual C++ Professional Edition, we will generate a computer programming to portray the 2-soliton solutions of Equation (2) and later produce various graphical outputs by using GnuPlot to produce various Encapsulated Postscript (eps) files. A simple computer program written to portray the 2-soliton solutions of the KP equation is given in the following subsection.

### 4.1 Computer Program for 2-KP Solitons

```
#include < iostream.h >
#include < fstream.h >
#include < math.h >
#define N 500

void main( )
{
      double k1, k2, A12, t, f, f1, f2, al, a2;
      double x[N + 1], u[N + 1], y;
      double n1, n2, l1, l2, ml, m2, wl, w2, Eta1, Eta2, Epsilon1, Epsilon2;
      int i;
      ofstream ofp;
      ofp.open("1.out",ios::out);
      n1 = 3; n2 = 3 + exp(–10); l1 = –2; l2 = 3; t = 1;
      k1 = l1 + n1;
      k2 = l2 + n2;
      a1 = 1;
```

```
    a2 = 1.3;
    m1 = n1*n1 – l1*l1;
    m2 = n2*n2 – l2*l2;
    w1 = –((3*ml*ml/k1) + (k1*k1*k1));
    w2 = –((3*m2*m2/k2) + (k2*k2*k2));
    Epsilon1 = a1/k1;
    Epsilon2 = a2/k2;
    A12 = ((l1 – l2)*(n1 – n2))/((l2 + n1)*(l1 + n2));
    y = –40;
    while (y< = 40)
    {
        x[0] = –50;
        for(i = 0; i< = N; i++)
        {
            Eta1 = k1*x[i] + m1*y – w1*t;
            Eta2 = k2*x[i] + m2*y – w2*t;
            f = 1 + Epsilon1*exp(Etal) + Epsilon2*exp(Eta2)
              + Epsilon1*Epsilon2*A12*exp(Eta1 + Eta2);
            f1 = k1*Epsilon1*exp(Eta1) + k2*Epsilon2*exp(Eta2)
              + (k1 + k2)*Epsilon1*Epsilon2*A12*exp(Eta1 + Eta2);
            f2 = k1*k1*Epsilon1*exp(Eta1) + k2*k2*Epsilon2*exp(Eta2)
              + (k1 + k2)*(k1 + k2)*A12*Epsilon1*Epsilon2*exp(Eta1 + Eta2);
            u[i] = 2*((f*f2 – f1*f1*)/(f*f));
            if(u[i] –0.05>0 | | u[i] + 0.05<0)
              ofp << x[i] << “  ” << y << “  ” << u[i] << “ ” <<
endl;
            x[i + 1] = x[i] + 0.16;
        }
        y+ = 1;
    }
    ofp.close();
}
```

### 4.2 Computer Simulations Full Resonance

For the full resonance, the values of $n_1$, $n_2$, $l_1$ and $l_2$ were chosen as below:

$$n_1 = 3, \quad n_2 = 3, \quad l_1 = -2, \quad l_2 = 3, \quad (A_{12} = 0) \tag{19}$$

By using these values, the computer simulation produces triad as shown in Figure 4.

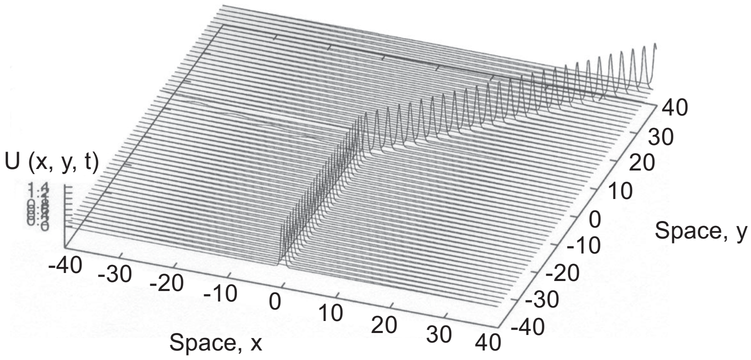


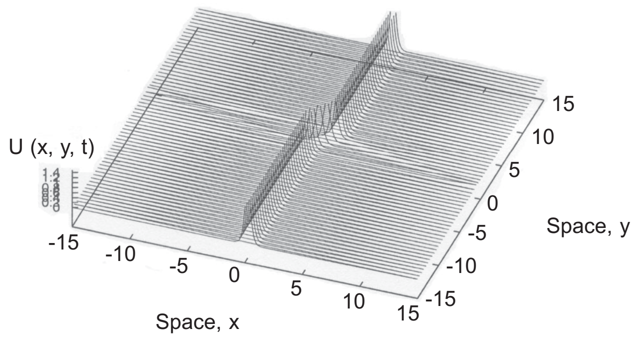

**Figure 4** 3D plot of a triad

## 4.3 Computer Simulations Partial Resonance

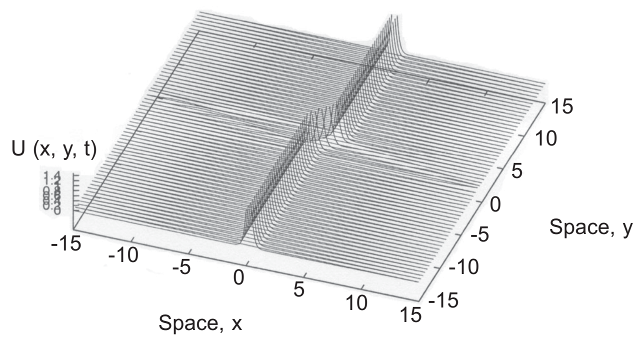
On the other hand, if we choose

$$n_1 = 3, \quad n_2 = 3 + \exp(-p), \qquad l_1 = -2, \quad l_2 = 3, \quad (A_{12} \approx 0) \tag{20}$$

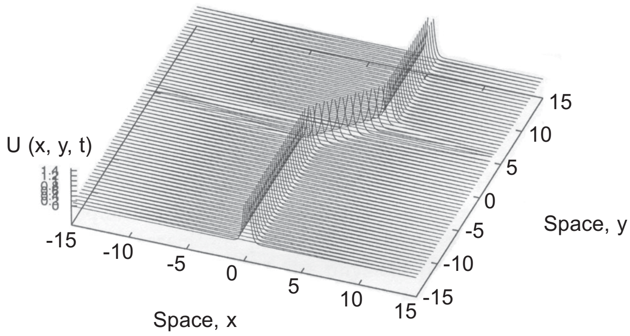
with $p = 20$ for the partial resonance, we will get a quadruplet representing the partial resonant soliton. The bigger the value of $p$ in $n_2$, the longer will be the resonant soliton $S_{23}$.




(a) $p = 10$




(b) $p = 20$




(c) $p = 30$

**Figure 5** A set of quadruplet with different values of $p$

## 4.4 Computer Simulations Non-Resonance

While for the non-resonance, we have

$$n_1 = 1, \quad n_2 = 3, \quad l_1 = 2, \quad l_2 = 3, \quad (A_{12} = 1) \tag{21}$$

The values of $l_1$ and $l_2$ are specifically chosen to ensure that the amplitude of solitons are positive. The amplitude of first soliton $S_1$ and second soliton $S_2$ were determined by $\frac{1}{2}k_i^2$ with $i = 1, 2$ and $k_i$ as defined in Equation (8). The value of $k_i$ must be positive to produce positive amplitude of the solitons. With this conditions, we will have a cross as shown in Figure 6.

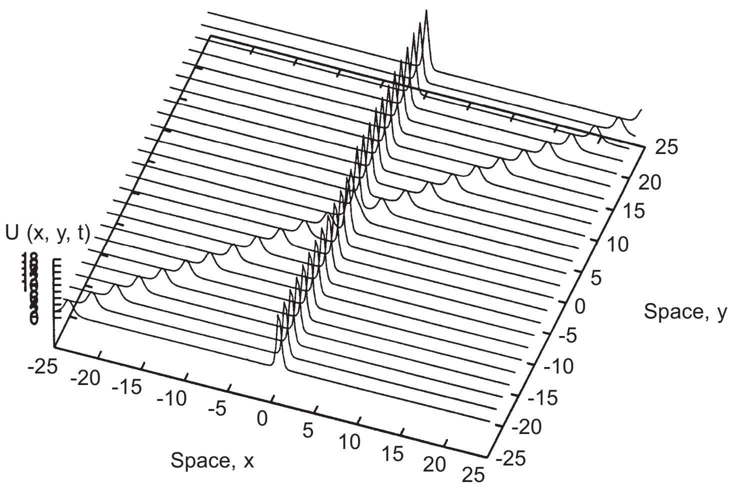



**Figure 6** 3D plot of a cross

## 5.0 CONCLUSION

From the computer simulations, we observed that there were three different resonance structures in two-soliton solutions of the KP equation which are full resonance, partial resonance and non-resonance structures. In partial resonance, the value of $A_{12}$ played an important role in determining the length of the resonant soliton $S_{23}$. In future, we intend to study the interactions patterns of three-soliton solutions in the KP equation and try to model some of the physical phenomena by using the KP equation in marine ocean basin.

## REFERENCES

[1] Ong, C. T. 1993. Various Aspects of Solitons Interactions. M.Sc. Thesis (Applied Mathematics). Universiti Teknologi Malaysia.

[2] Anker, D., and N. C. Freeman. 1978. Interpretation of Three-Soliton Interactions in Terms of Resonant Triads. *Journal of Fluid Mech*. 87(1): 17-31.

[3] Miles, J. W. 1977. Obliquely Interacting Solitary Waves. *Journal of Fluid Mech*. 79(1): 157-169.

[4] Miles, J. W. 1977. Resonantly Interacting Solitary Waves. *Journal of Fluid Mech*. 79(1): 171-179.

[5] Satsuma, J. 1976. N-Soliton Solution of the Two-Dimensional Korteweg-de Vries Equation. *Journal of Physical Society Japan*. 40(1): 286-290.

[6] Hirota, R. 1971. Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys. Rev. Lett*. 27(18): 1192-1458.