ISOGEOMETRIC ANALYSIS OF PLANE STRESS STRUCTURE

CHUM ZHI XIAN

A project report submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Engineering (Civil-Structure)

Faculty of Civil Engineering
Universiti Teknologi Malaysia

JUNE 2015

Specially dedicated to my beloved parents, brother, sister, lecturers, and friends.

ACKNOWLEDGEMENTS

First of all, I would like to express gratitude to my supervisor, Dr. Mohd. Ridza bin Mohd. Haniffah and co-supervisor, Dr. Airil Yasreen Mohd. Yassin for their patience, guidance and support as well as time that have been contributed throughout this research study. Also, special thanks to Dr. Ahmad Razin bin Zainal Abidin @ MD. Taib who has worked out this project together.

I am also thankful to Al Akhbar Mohd. Nor and Mohd. Zhafri Jamil. Without their helping hand, I would not be able to achieve this far. Last but not least, I would like to express my appreciation to my friends and family that have given me direct and indirect support in this project.

ABSTRACT

Differential equations are derived to describe the physical phenomena in engineering system. In this project, differential equations of simple rectangular plane stress problem were first derived and solved using Isogeometric Analysis (IGA) and Finite Element Method (FEM). The root idea of IGA is to use a single basis to represent the geometry and the analysis fields in order to overcome the bottleneck in Computer Aided Design (CAD) and Computer Aided Engineering (CAE). The aim is to investigate the performance of IGA as compared to FEM. It is realized that the main difference between the two numerical techniques adopted is the formulation of shape functions. Therefore, emphasis is put on the formulation of IGA using Non-Uniform Rational B-Splines (NURBS) as the basis function where the results obtained are compared against finite element formulation which uses polynomial functions for the shape functions. Besides that, the results by both formulations are verified against exact solution and commercial software. Although only the shape function differs, IGA uses a global shape function over the domain while FEM uses the same local shape functions over the elements in the domain. Performance study on IGA was also carried out. It has been found that the convergence of IGA is comparable to conventional FEM and the error is small against the exact solution. Despite more time is needed to compute the shape functions in IGA, there are various refinement mechanisms in IGA where knot insertion shows the best performance in this study. In short, IGA is worthwhile to be used as an analysis tool to initiate the communication between computer aided design (CAD) and computer aided engineering (CAE).

ABSTRAK

Persamaan terbitan diperolehi untuk menggambarkan fenomena fizikal dalam sistem kejuruteraan. Dalam kajian ini, persamaan terbitan bagi masalah tegasan dalam satah yang mempunyai bentuk segiempat diperolehi terlebih dahulu dan kemudiannya diselesaikan dengan menggunakan *Isogeometric Analysis* (IGA) dan Kaedah Unsur Terhingga (FEM). Tujuannya adalah untuk menyiasat prestasi IGA berbanding dengan FEM. Perbezaan utama antara kedua-dua kaedah berangka ini adalah pada penggubalan fungsi bentuk (*shape functions*). Oleh itu, penekanan diletakkan dalam penggubalan IGA yang menggunakan *Non-Uniform Rational B-splines* (NURBS) sebagai fungsi asas (*basis function*) di mana hasil yang diperolehi akan dibandingkan dengan FEM yang menggunakan polinomial sebagai fungsi bentuknya. Selain itu, hasil kajian daripada kedua-dua formulasi ini telah dibandingkan dan disahkan dengan penyelesaian analitikal dan penggunaan perisian komersial. Walaupun hanya fungsi bentuk yang berbeza, IGA menggunakan fungsi bentuk yang global merangkumi keseluruhan domain manakala FEM menggunakan fungsi bentuk yang spesifik kepada satu elemen dan ianya adalah sama untuk keseluruhan elemen di dalam domain. Berdasarkan kepada kajian prestasi IGA, didapati bahawa penumpuan (*convergence*) IGA adalah setanding dengan FEM dengan ralat yang kecil berbanding dengan penyelesaian analitikal. Walaupun lebih banyak masa diperlukan untuk mengira fungsi bentuk IGA, terdapat pelbagai mekanisme untuk memperhalusi prestasi IGA dimana *knot insertion* menunjukkan prestasi yang terbaik dalam kajian ini. Sebagai rumusan, IGA boleh digunakan sebagai alat analisis bagi memulakan komunikasi di antara reka bentuk bantuan komputer (CAD) dan kejuruteraan bantuan komputer (CAE).

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS / TERMINOLOGY

CAD	Computer Aided Design
CAE	Computer Aided Engineering
FEA	Finite Element Analysis
FEM	Finite Element Method
IBP	Integration By Part
IGA	Isogeometric Analysis
NURBS	Non-Uniform Rational B-Splines
PDE	Partial Differential Equation
UDL	Uniformly Distributed Load
WRD	Weighted Residual Method

LIST OF SYMBOLS

a	-	Length of plane stress element in *x*-direction
b	-	Length of plane stress element in *y*-direction
$\boldsymbol{B}_{i,j}$	-	Control net
C	-	Convergence rate
e	-	Relative error
E	-	Young's modulus
F_x	-	Body force in *x*-direction
F_y	-	Body force in *y*-direction
m	-	Multiplicity
M	-	Shape function in *y*-direction
N	-	Shape function in *x*-direction
p	-	Polynomial order in *x*-direction
q	-	Polynomial order in *y*-direction
t	-	Thickness
u	-	Deformation in *x*-direction
v	-	Deformation in *y*-direction
$[E]$	-	Material properties

$[k]$	-	Element stiffness matrix
$[K]$	-	Global stiffness matrix
$[\partial]$	-	Differential operator matrix
$\{N\}$	-	Shape functions vector
$\{r\}$	-	Force vector
Ω	-	Original domain
Ω_e	-	Finite element domain
Ω_h	-	Domain formed by assemblage of elements
σ_{xx}	-	Normal shear stress in *x*-direction
σ_{xy}, σ_{yx}	-	Shear stress
σ_{yy}	-	Normal shear stress in *y*-direction
ε_{xx}	-	Strain in *x*-direction
ε_{xy}	-	Shear strain
ε_{yy}	-	Strain in *y*-direction
ω	-	Weighting
υ	-	Poisson's ratio
ξ	-	Knot vectors in *x*-direction
η	-	Knot vectors in *y*-direction

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Elasticity is a part of solid mechanics that deals with stress and deformation of solid continua. There are two categories of plane elasticity; plane stress and plane strain. However, the interest is on plane stress structure and hence only plane stress will be brought into further discussion. Plane stress element is a two-dimensional solid and is used to model thin body or structure that is subjected to in plane loading (or boundary stresses). Plane stress solids are solids whose thickness in the z direction is insignificant, less than one-tenth compared to the smallest dimension in the x and y direction (Hutton, 2004). In practical, plane stress is used to model structures such as shear walls, load bearing walls and steel web.

The study of plane stress formulation is important not only for its direct application to physical problems but as a basis that is used in other elements formulations. For example, plane stress formulation can be evolved to Mindlin's plate formulation. Meanwhile, in the discussion of shell elements, its formulation can be viewed as combination of plane stress and plate element. Also, with some modifications, plane stress element can be used to model fluid flow in the field of fluid dynamics. The differential equations of plane stress element can be derived using principle of conservation of linear and angular momentum and solved numerically. Myriad numerical techniques are commonly used to solve differential equations such as Finite Difference Method, Finite Element Method (FEM) and Meshfree method. In this study, Isogeometric Analysis (IGA) and FEM are adopted to solve the plane stress problem.

The development of Finite Element Method is one of the most advanced in the field of numerical methods and was introduced in 1950s. It is a numerical technique used to solve differential equations. Generally, in FEM, a complicated shape continuum is divided into elements; finite elements and the individual elements are then connected together by a mesh. It uses weighted residual method and interpolation function to construct the shape function. FEM is widely used in engineering field because of its versatility for complex geometry and flexible for many types of linear and non-linear problems. There are plenty of well-developed FEM software packages built to solve most of engineering problems related to solids and structures. Nevertheless, FEM has its drawbacks and limitations. For instances, analyst has to spend most of the time in mesh creation and is required to recover the accuracy of stresses in post processing stage, possess difficulty in adaptive analysis in ensuring high accuracy and limitation in analyzing of problems under large deformation, crack and simulating breakage of material (Liu, 2010).

Besides that, the existence of gap between the current technology and engineering process will require designers to draw their drawing in Computer Aided Design (CAD) file and then translated into Computer Aided Engineering (CAE) by the engineers. The concept of analysis procedure based on CAD is referred to Isogeometric Analysis. Isogeometric Analysis seeks to unify the field of CAD and numerical analysis such as FEM and Meshfree, hence bridging the gap of CAD and CAE. Among the computational geometry technologies used in IGA, Non-Uniform Rational B-Splines (NURBS) is most widely used in engineering design. The preeminence of NURBS in engineering design as compared to other computational geometry technologies is generally because of the convenient for free-form surface modeling (J.A. Cottrell, *et al.*, 2009). For example, it can represent exactly all conic sections and there are many efficient and numerically stable algorithms to generate

NURBS. Since NURBS is the most popular computational geometry in CAD, it is selected to be the basis function in the derivative of domain equation and analysis.

FEM and IGA start to diverge during the construction of shape functions. In FEM, the shape functions are constructed using polynomial interpolation functions and the shape functions are the same for the entire element. On the other hand, the shape functions constructed in IGA are based on knot vector of the patch of the entire domain. This means that the NURBS parameter space is local to patches rather than local to elements in Finite Element Analysis (FEA). A more detailed description of NURBS is explained in Section 2.4.2. Both methods follow the same procedure once the global discretized system equation is established. The formulations of both numerical methods are further discussed in Chapter 3.

Realizing that this study is in its preliminary phase i.e. in the realm of basics and fundamental of IGA, only plane stress problem is going to be solved. This study focuses on the formulation of IGA and FEM to solve plane stress problem and verifying the results against exact solutions. Besides that, commercial software, COMSOL is used to check the validity of overall deflection behavior.

1.2 Problem Statement

There is a shortcoming of current technology and engineering process where the designers have to generate their drawings in computer aided design (CAD) files and then translated by engineers into analysis-suitable geometries, meshed and input to large-scale numerical analysis codes (Hughes *et al.*, 2005). This reflects the existence of communication gap between CAD and CAE. Thus, the motivation of this study is to fill the gap of CAD and CAE and also reduce work redundancy. At this preliminary stage of study and in order to initiate the conversation between CAD and numerical analysis, it is of interest to:

1. Solve plane stress structure using Isogeometric Analysis and Finite Element Method.

2. Report the performance of Isogeometric Analysis against Finite Element Method.

1.3 Objectives

The objectives of this study are as follows:

1. To formulate Isogeometric Analysis and Finite Element Analysis for plane stress problem.

2. To verify the formulations with closed form solution.

3. To assess the performance of Isogeometric Analysis against Finite Element Method.

The programming of both methods of IGA and FEM to achieve the objectives will be done using MATLAB.

1.4 Scope of Study and Limitation

The scope of the study and limitations were listed in the following:

1. There are a number of candidate computational geometry technologies used in Isogeometric Analysis such as S-patches, A-patches, T-spline and NURBS. Only NURBs will be used as basis function throughout the analysis.

2. The problem to be solved is a cantilever rectangular plane stress with boundary condition fixed at one end.

3. The loading on the plane stress is uniformly distributed load applied on the other end of plane stress, opposite of the constraint end. No other load pattern is considered.

4. The plane stress structure is analyzed based on linear analysis only.

1.5 Significance of the Study

The usage of IGA to solve partial differential equations is relatively new. Hence, more studies on IGA is sought to establish the robustness and performance of this numerical technique. By making use the basis in CAD technology for analysis, results is ought to be more accurate as the basis represents the actual geometry of domain. In fact, it can be used to bridge the gap between CAD and CAE and reduce the cost of analysis.

1.6 Outline of Thesis

Theoretical background of plane stress structure will be discussed in Chapter 2. Besides that, numerical techniques of interest, FEM and IGA are reviewed and related works that have been done will be discussed into detail. Chapter 3 is the methodology which explains in detail of the procedure in the derivation of the differential equation of plane stress element. Formulations of plane stress using FEM and NURBS based IGA are described in this chapter. Meanwhile, Chapter 4 shows the result of deflection at one corner of the plane stress element obtained from both FEM and IGA. The results obtained are compared against exact result and critical discussion is made in this section. Lastly, the summary of the whole thesis is made in Chapter 5 with inclusive of recommendation for future research in this topic.

REFERENCES

Airil, Y., Akhbar, A., Kasiman, E. H. and Jamil, Z. (2013) *Linear and Nonlinear Finite Element Analysis for Solids and Fluids*. Unpublished Note. Malaysia.

Bathe, K. J. (1996) *Finite Element Procedures*. (3rd ed.). Texas: Prentice Hall.

Chien, T. H., Dung, T. T., Canh, L. V. and Hung, N. X. (2012) *Isogeometric Limit Analysis for Plane Stress Problems.* The International Conference on Advances in Computational Mechanics (ACOME), Ho Chi Minh City, Vietnam.

Cottrell, J. A., Hughes, T. J. R. and Bazilevs, Y. (2009) *Isogeometric Analysis: Toward Integration of CAD and FEA.* Chichester: John Willey & Sons, Ltd.

Hughes, T. J. R., Cottrell, J. A. and Bazilevs, Y. (2005) *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement.* Austin: Science Direct.

Hutton, D. V. (2004). *Fundamental of Finite Element Analysis*. New York: McGraw-Hill.

Liu, G. R. (2010). *Meshfree Methods, Moving Beyond the Finite Element Method.* (2nd ed.). London: CRC Press LLC.

Kacprzyk, Z. and Trybocki, Z. (2013) *Isogeometric Plane Stress Analysis.* Computer Methods in Mechanics, Poland.

Reddy, J. N. (2006) *An Introduction to The Finite Element Method*. Texas: McGraw-Hill.