GLOBAL OPTIMIZATION USING HOMOTOPY WITH 2-STEP PREDICTOR-CORRECTOR METHOD

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To My Beloved Family and Friends

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ABSTRACT

In optimization, most established search methods are local searches. Thus the development of a method that can be relied upon to find global solutions are therefore highly significant. Homotopy Optimization with Perturbations and Ensembles (HOPE) is such a method. In HOPE, a large storage space is required to store the points generated during its execution and subsequently its space and time complexity will become higher which causes the operational cost of HOPE to be expensive. This is the weakness of HOPE. In this study, a new method which is an improvement over HOPE called Homotopy 2-Step Predictor-Corrector Method (HSPM) is proposed. HSPM applies the Intermediate Value Theorem (IVT) coupled with the modified Predictor-Corrector Halley method (PCH) to overcome the weakness of HOPE. In HSPM, subintervals within which the extremizers lie, called 'trusted' intervals are found based on IVT. A random point is selected from the 'trusted' interval as an initial point to a local search. Each 'trusted' interval produces one local solution. Lastly, the global solution is determined from the local solutions accumulated. From the results, HSPM has been very successful as a minimization tool. It is able to cope with various types of functions' landscapes and able to detect more than one global solutions. Furthermore, HSPM can identify all the minimizers regardless of the step sizes used by the homotopy function. Hence, it has a high success rate in getting to a global minimizer compared to HOPE. Complexity analysis is employed to show the improvement achieved by HSPM. Based on the analysis, HSPM successfully managed to reduce the computational burden suffered by HOPE and acts as a good method in solving one dimensional optimization problem. However, to cope with the requirements today it needs to be extended to deal with multivariable functions for its future work.

ABSTRAK

Dalam pengoptimuman, kebanyakan kaedah pencarian yang wujud masa kini adalah kaedah pencarian setempat. Oleh itu, pembangunan suatu kaedah yang diyakini untuk memperoleh penyelesaian sejagat adalah sangat bererti. Pengoptimuman Homotopi dengan Usikan dan Ensembel (HOPE) adalah seumpamanya. HOPE memerlukan ruang simpanan yang besar untuk menyimpan titik terhasil semasa pelaksanaannya, akibatnya kekompleksan ruang dan masa menjadi tinggi yang menyebabkan kos pelaksanaan HOPE mahal. Ini adalah kelemahan HOPE. Dalam kajian ini, suatu kaedah baharu yang merupakan penambahbaikan kepada HOPE, dikenali sebagai Kaedah Homotopi dengan Peramal-Pembetul 2-Langkah (HSPM) dikemukakan. HSPM mengaplikasikan Teorem Nilai Pertengahan (IVT) diganding dengan kaedah Peramal-Pembetul Halley (PCH) untuk mengatasi kelemahan HOPE. Dalam HSPM, subselang di mana pengekstremum berada dikenali sebagai selang 'boleh-percaya' diperoleh berdasarkan IVT. Suatu titik rawak dipilih daripada selang 'boleh-percaya' sebagai titik permulaan untuk gelintaran setempat. Setiap selang 'boleh-percaya' akan menghasilkan satu penyelesaian setempat. Akhirnya, penyelesaian sejagat akan ditentukan daripada penyelesaian setempat terkumpul. Berdasarkan keputusan, HSPM sangat berjaya sebagai alat peminimuman. Ia mampu menangani pelbagai jenis landskap fungsi dan boleh mengesan lebih daripada satu penyelesaian sejagat. Tambahan pula, HSPM boleh mengenal pasti semua peminimum tanpa mengira saiz langkah yang digunakan oleh fungsi homotopi. Oleh itu, ia mempunyai kadar kejayaan yang tinggi untuk sampai ke peminimum sejagat berbanding HOPE. Analisis kekompleksan digunakan untuk menunjukkan penambahbaikan yang dicapai oleh HSPM. Berdasarkan analisis, HSPM berjaya mengurangkan beban pengiraan yang dialami oleh HOPE dan berperanan sebagai kaedah yang baik untuk menyelesaikan masalah pengoptimuman satu matra. Walau bagaimanapun, untuk menangani keperluan semasa, ia perlu diperluaskan pada masa hadapan supaya dapat menyelesaikan masalah melibatkan fungsi berbilang pembolehubah.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

DAEs	-	Differential Algebraic Equations
GO	-	Global Optimization
GOM	-	Global Optimization Methods
HR	-	Hit and Run method
HOPE	-	Homotopy Optimization with Perturbations and Ensembles
HAM	-	Homotopy Analysis Method
HCM	-	Homotopy Continuation Method
HOM	-	Homotopy Optimization Method
HPM	-	Homotopy Perturbation Method
HSPM	-	Homotopy 2-Step Predictor-corrector Method
IVT	-	Intermediate Value Theorem
M8	-	Mathematica version 8.0
min	-	Minimize/ minimum
max	-	Maximize/ maximum
ODEs	-	Ordinary Differential Equations
PCH	-	Predictor-Corrector Halley's method
sup	-	Supremum
SCA	-	Space Complexity Analysis
TCA	-	Time Complexity Analysis

LIST OF SYMBOLS

Ω	-	A closed set which contained solution, big-Omega notation
K	-	A corresponding value of *c* over function *f*, constant
$HOPE(x)$, $HSPM(x)$	-	A function
D	-	A nonempty, closed set
c	-	A value/ element in between interval $[a,b]$
$g(x)$	-	Auxiliary function
O	-	Big oh notation, constant
θ	-	Big theta notation
L,P	-	Constant
k	-	Constant, number of iteration
m	-	Controller of the sharpness of valley of Eq (4.7), parameter to adjust the step size *st*
h	-	Convergence-control parameter
$c_i, i = 1, 2, ...$	-	Cost used for an algorithm
■	-	End of proof
x^*	-	Extremizers
$f(x)$	-	Function *f* or target function
$x^{(1)}$	-	Global solution
$H(x,\lambda)$	-	Homotopy function
λ	-	Homotopy parameter
$subinterval_i$	-	i number of subintervals within interval $[a,b]$

ε	-	Infinitesimal change of the dependent variable, error, stopping criterion
δ	-	Infinitesimal increment of the independent variable
I_n	-	Interval
x_0 , x^0	-	Initial guess
$x_{i,0}$	-	Initial point, randomly select from $subinterval_i$
$x_j^{(k-1)}$	-	j^{th} point in the ensemble at the start of iteration *k*
α	-	Least upper bound for *A*
a	-	Lower bound of an interval
a_n	-	Lower bound of interval I_n
a_i	-	Lower bound of $subinterval_i$
$c_{\max}$	-	Maximum number of points in an ensemble
M	-	Maximum number of $subinterval_i$, constant
mpt	-	Midpoint of interval
d	-	Non-empty subset of $\mathbb{R}$
n, *j*, *i*	-	Number of iteration
$\hat{c}$	-	Number of perturbations generated of each point in the ensemble
$c^{(k-1)}$	-	Number of points in the ensemble at the beginning of iteration *k*
-ve	-	Negative value
$F(x)$	-	Objective function
ξ	-	Perturbation
$x_{j,0}^{(k)}$	-	Point found by minimization starting at $x_j^{(k-1)}$
$x_{j,i}^{(k)}$	-	Point found by minimization starting at the i^{th} perturbation of $x_j^{(k-1)}$
+ve	-	Positive value
X, *Y*	-	Set
$O(g(n))$, $O(h(n))$,	-	Set of complexity function

A	-	Set contaning element lower bound of I_n
n	-	Size input, number of iteration, constant
$\vartriangle\lambda$, $\Delta^{(k)}$, λ, s, st	-	Step length/ step size
x^k	-	Solution computed by local search
$SB_{HOPE}(n)$	-	Space complexity of HOPE in best case
$SW_{HOPE}(n)$	-	Space complexity of HOPE in worst case
$SB_{HSPM}(n)$	-	Space complexity of HSPM in best case
$SW_{HSPM}(n)$	-	Space complexity of HSPM in worst case
c^3	-	Three-times-continuously differentiable
$TW_{HOPE}(n)$	-	Time complexity of HOPE in worst case
$TB_{HOPE}(n)$	-	Time complexity of HOPE in best case
$TW_{HSPM}(n)$	-	Time complexity of HSPM in worst case
$TB_{HSPM}(n)$	-	Time complexity of HSPM in best case
c^2	-	Twice-continuously differentiable
b	-	Upper bound of an interval
b_n	-	Upper bound of interval I_n
b_i	-	Upper bound of $subinterval_i$
x	-	Variable
$\lambda^{(k)}$	-	Value of parameter homotopy in iteration k

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction

This chapter is providing the definition and brief explanation to give a basic and clearer understanding for central idea of this study and it is structured as follows: an introduction of this study optimization is described in Section 1.2 and the background of the problem is given in Section 1.3 while statement of the study is shown in Section 1.4. Next, we discuss the objectives, scope and significance of this study in Section 1.5, Section 1.6 and Section 1.7 respectively. A research outline is provided at the last section of the chapter.

1.2 Optimization

What is optimization? Optimization is a mathematical method to solve decision making problem. It consists of an objective function, $F(x)$ which needs to be maximized or minimized to find its extremizer or the best solution of that function.

Optimization had been used widely in our daily life. For example, optimization can be used in engineering design, it helps in choosing the design parameters to improve some objectives. It also is useful in data analysis by extracting model parameters from data while minimizing some error measures. Optimization also plays an important role in business decision, it is a necessary tool to help businessman to vary decision parameters to maximize profit hence minimize cost.

A function can have more than one extremizers, if it gives the function a minimum value, we call a minimizer while if it gives the function a maximum value, we call a maximizer. Since a maximum optimization problem can convert to a minimum optimization problem, thus in this study we will concentrate on the minimization problem.

Generally finding a global solution in an optimization involves two stages, which are local search stage and global search stage. In local search stage, we will find the local solution to the given function by using iterative methods such as Newton's method, Quasi-Newton method, and Conjugate Gradient method. There are many method for solving global optimization problem such as Filled Function method, Tunnelling method, Heuristic methods, and Homotopy Optimization with Perturbations and Ensembles (HOPE) method. A function can have more than one local solution, but not all local solutions are global. Global search stage is a process to identify the global solution from the local solutions. A global solution will give the function the lowest value among the local solutions in the given function.

1.3 Background of Problems

Homotopy is a basic concept in Topology. The basic idea underlying a homotopy method is to deform a simple solvable problem continuously into the given (hard to solve) problem, while solving a continuous sequence of deformed problems (Arkowitz, 2010).

Homotopy Continuation Method (HCM) was introduced to solve the problems of nonlinear optimization and also systems of nonlinear equations (Allgower and George, 1990). This method deforms a simple function into the function of interest by tracing path, computes series of zeros and ends in zero of that function of interest. Since the homotopy methods converge to a solution for any arbitrary chosen initial condition, they are said to be globally convergent.

Dunlavy and Leary (2005) proposed a homotopy method to solve optimization problem called Homotopy Optimization Method (HOM), this method can solve optimization problem without path-tracing. They also introduced a new method called Homotopy Optimization with Perturbation and Ensembles (HOPE). HOPE is a method which allows HOM to follow an ensemble of points obtained by perturbation of previous one. In other words, we can say that HOPE is an extension of HOM.

In the work of Dunlavy and Leary (2005), points in the resulting ensemble are perturbed in various directions and used as starting points to find the minimizers of homotopy function as it deforms the simple solvable function into the target function. In each iteration, the ensemble members carried forward the previous iteration were perturbed and used as starting points to find other minimizers, and hence it produced a lot of calculation works in its inner loops. Thus, the cost of this algorithm will increase when the ensemble members increase. In order to overcome this problem, we proposed a new method by improve HOPE method with the Intermediate Value Theorem (IVT) and Predictor-Corrector Halley's method (PCH).

1.4 Problems Statements

This research will embark on the development of a new method, a modification of HOPE to yield an algorithm which has a lower time complexity than the existing HOPE and also have a higher probability in getting to a global solution than HOPE

1.5 Objectives of the Study

The main objectives of this research are:

i. To apply the homotopy concept as a numerical tool in order to find the global solution of nonlinear equations.
ii. To identify a method which can find the "trusted" interval within which extremizers lie.
iii. To develop an algorithm by modifying the HOPE with the concept of IVT and PCH method.
iv. To establish the efficiency of new developed method over HOPE.

1.6 Scope of the Study

In this study, the new developed method is designed to deal with one dimensional optimization problem and Predictor-Corrector Halley method will be used as a local search method to find the solution of the problem. The functions used in this research are at least three-times-continuously differentiable, C^3 function over a closed interval. The efficiency of the method proposed will be measured based on the time and space complexity analyses. Furthermore, a comparison between HOPE and the new developed method based on the percentage of success in arriving to the global solution will be shown in this study.

1.7 Significance of the Study

The findings of this study will enlarge the knowledge in the usage of homotopy to overcome the problem in finding the local solution such as divergence and bifurcation. The method proposed here can be used in solving global optimization problem. This more efficient method can be useful in data analysis, business, and so on.

1.8 Research Outline

This research consists of four chapters and the contents of each chapter is described as follow:

Chapter 1 is related to the introduction of the topic of research. The contents of this chapter includes background of the problem, statement of the problem, objectives of the study, scope of the study and the significance of the study as well.

Chapter 2 is about the literature review on this research. The previous and recent studies on this research area are reviewed and discussed. The information from the materials such as journal will be stated.

Chapter 3 included the overall research framework and methodology. The techniques applied to complete research objectives are described. Chapter 4 is about the description of functions chosen for the numerical experiments and also its respective results, a trade-off analysis based on the time and space complexity, and a comparison for the success rate to getting a global solution in between HOPE and HSPM.

Lastly, a summary for the outcomes of each chapter, conclusion of the research and the suggestion for future works to extend this study will be given in Chapter 5.

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