SOLVING TWO-DIMENSIONAL GROUNDWATER FLOW EQUATION USING ALTERNATING DIRECTION IMPLICIT METHOD

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To my beloved mother, Suziana binti Sidek, my father, Ahmad Nordin bin Abdullah,
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Thank you for all of your greatest support and everlasting love

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ABSTRACT

Groundwater model can be described as a mathematical model and the equation of groundwater is governed by partial differential equation. In order to solve the groundwater flow equation, numerical method such that Finite Difference Method (FDM) is used. In this research, a two-dimensional transient groundwater flow equation for a confined, nonleaky, and homogeneous with mixed boundary conditions is solved using Alternating Direction Implicit (ADI) method where ADI method is one of the FDM. The algorithm of ADI method has been developed for three different types of boundary conditions that is Dirichlet condition, Neuman condition and Mixed condition. The transient groundwater flow equation has been derived and was solved using ADI method by Matlab software. Then, the results obtained were compared to analytical solution. Since the solutions from numerical method provide the small error when compared to the analytical solutions, it therefore can be concluded that ADI method provides good approximations in solving two-dimensional groundwater transient flow problem.

ABSTRAK

Model air bawah tanah boleh digambarkan sebagai model matematik dan persamaan air bawah tanah diwakilkan oleh persamaan pembezaan separa. Bagi menyelesaikan persamaan aliran air bawah tanah, kaedah berangka digunakan iaitu Kaedah Beza Terhingga. Dalam kajian ini, persamaan aliran air bawah tanah dua dimensi yang terbatas, tiada kebocoran, dan homogen dengan syarat bercampur diselesaikan menggunakan kaedah Lelaran Tersirat Berarah (LTB). Kaedah ini merupakan salah satu kaedah beza terhingga. Algoritma kaedah Lelaran Tersirat Berarah telah dibangunkan untuk tiga jenis syarat sempadan iaitu syarat Dirichlet, syarat Neumann dan syarat bercampur. Persamaan aliran air bawah tanah mantap telah diperolehi dan telah diselesaikan dengan kaedah LTB serta perisian Matlab. Kemudian, keputusan yang diperolehi dibandingkan dengan penyelesaian analisis. Oleh kerana penyelesaian berangka memberikan ralat yang kecil apabila dibandingkan dengan penyelesaian analitikal, kaedah Lelaran Tersirat Berarah dapat memberikan penyelasaian yang baik bagi menyelesaikan masalah aliran dua dimensi air bawah tanah.

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LIST OF ABBREVIATIONS

PDE - Partial Differential Equation

FDM - Finite Difference Method

ADI - Alternating Direction Implicit

AGE - Alternating Group Explicit

FEM - Finite Element Method

SSOR - Symmetric Successive Over Relaxation

CPU - Central Processing Unit

IADI - Iterative Alternating Direction Implicit

IRBFE - Integrated Radial Basis Function Element

MMPDE - Moving Mesh Partial Differential Equation

NAPL - Nonaqueous Phase Liquid

REV - Representative elementary volume

LIST OF SYMBOLS

h - Hydraulic head

S - Storability

T - Transmissivity

t - Time

u - Dependent variable

x, y - Independent variables

a, b, c, d, e, f, g, r - Parameters

i, j - Indices

k - Iterations parameter

Δx , Δy , Δt - Size of interval at each axis

A, $\mathbf{h}$, $\mathbf{b}$ - Matrix

Γ - Boundary

Q - Rate of flow

Δh - Head difference

l ,L - Length of the flow path

K - Hydraulic head

V, q - Specific discharge

α - Porosity

A - Cross-sectional area

ρ_w - Fluid density

M - Mass flux

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Most of the problem can be expressed mathematically known as mathematical problems. The mathematical problems can be solved using two different methods. There are analytical methods and numerical methods. However when the problem is suffered by complex region or the boundary condition is time dependent, the analytical method may be unsuccessful. Hence, the numerical method will be very useful because it can provide convenient method to obtain solution to mathematical problems.

It is clear that many important scientific problems are governed by partial differential equations (PDEs). PDEs form the basis of mathematical model which are related to phenomena of physical, chemical, biological, hydrological etc. Solving PDEs by means of numerical method is very crucial since most of the PDEs for practical problem cannot be solved analytically due to the restrictive condition.

Finite Difference Method (FDM) is one of the most popular numerical techniques. It is based on the approximations solution to the differential equations by using finite difference equations. This method provides a rationale for operating on the differential equations that make up a model and for transforming them into a set of algebraic equations. It also related to the grid points [1].

In this project, finite difference method that is Alternating Direction Implicit (ADI) method is used for solving two dimensional groundwater flow equation.

1.2 Background of Study

Groundwater is water located under the ground surface in soil pores and in the fractures of rock formations [2]. An aquifer is a geological unit such as rock or an unconsolidated deposit that can store and transmit the amount of water. There are two types of aquifers. Aquifers that are closed to the land surface, with continuous layers of high intrinsic permeability through the land surface to the base of aquifer known as unconfined aquifers or water table. Meanwhile, confined aquifers are overlain by a confining layer.

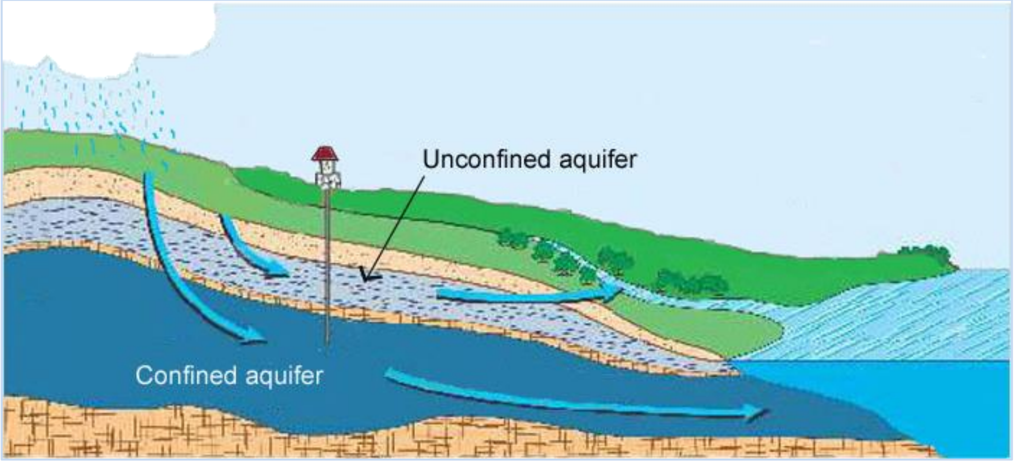

Figure 1.1: Groundwater flow and aquifers

The application of groundwater problems can be described in the mathematical models that derived from Darcy’s Law. Mathematical models of groundwater flow have been in use since late 1800s. The groundwater model is used to calculate the rate and movement of groundwater through the aquifer and confining units in the subsurface. The models are also importance in the decision making process of water resources system. Besides, they may also be used in some prediction of some future groundwater flow.

In general, a mathematical model for groundwater problems is governed by PDEs including specification of system geometry, boundary conditions and initial conditions for transient process. This problem can be solved by using analytical method and numerical method. In the past, analytical methods were commonly used for groundwater flow problems. An analytical solution of the PDE was brought up for a particular problem together with initial and boundary condition. The advantage of an analytical solution is that it usually gives an exact solution to the governing equation.

Many analytical solutions have been constructed for the flow equation, but most application are limited to well hydraulics problems involving radial symmetry [3][4][5]. Simplifying must always be made in order to construct a model. This is because the field situations are very complex to be simulated exactly. However, the assumptions needed to solve the mathematical model analytically are quiet restrictive. For example, the application of analytical methods is limited to simple problem such as analytical solutions require that the medium be homogeneous and isotropic.

In order to deal with more complex groundwater problem, the PDEs can be approximated numerically which is the continuous variables are replaced with discrete variables that are defined at grid blocks or nodes. Since the 1960s, numerical methods have been preferred for studying groundwater flow problems when the high-speed digital computers become available. There are two well-known numerical methods for solving the groundwater flow equation which are finite difference methods (FDM) and finite element methods (FEM). These numerical techniques require that the area of interest be subdivided into a number of smaller subareas called cells or elements associated with node points [6]. Both of these numerical approaches have their own advantages and disadvantages.

Generally, FDM are simpler conceptually and mathematically and are easier to program for a computer [7]. In this research, finite difference techniques such that Alternating Direction Implicit (ADI) method will be used in order to solve the two-dimensional groundwater flow.

1.3 Problem Statement

Groundwater flow in two dimensional can be either in steady-state or unsteady-state and also known as transient. Through this project, two-dimensional transient groundwater flow will be considered. Thus, finite difference method will be developed to approximate the solutions of equations in the form of

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t} \tag{1.1}$$

$$0 \le x \le a\,,\ 0 \le y \le b\,,\ t \ge 0$$

where x and y are the distance coordinates, t is time, h is a hydraulic head, S is the storativityy and T is a transmissivity. Equation (1.1) is used to find $h\left(x,y,t\right)$ with the given initial and boundary conditions. Hence, does ADI method can solve two dimensional groundwater flow?

1.4 Objective of the Study

The objectives of this study are:

1) To derive two-dimensional groundwater flow equation.
2) To solve two-dimensional groundwater flow using Alternating Direction Implicit (ADI) method by using Matlab software.
3) To analyzing the performance of Alternating Direction Implicit (ADI) method.

1.5 Scope of the Study

This project will focus on solving two-dimensional transient groundwater flow equation by using finite difference method. The specific approach used is ADI method.

1.6 Significance of the Study

The result of this study will be used as alternative approach to solve two-dimensional groundwater flow equation. This study will also leads to further study to solve problem related to parabolic equation by applying ADI method. Other than that, the readers will be able to derive the algorithm of this method as well as deriving the two-dimensional groundwater flow equation.

Besides, it is also important to the agriculturist, environmentalist, hydrologist etc since they used the groundwater flow model to predict the future groundwater flow, evaluate the impact assessment required for water in a regulated aquifer system as well as used the models as a tools for decision making in the water resource system management.

1.7 Organization of the Study

This report consists of five chapters. Chapter 1 provides an introduction to the research study as an overview of the current research. It consists of the background of the study, problem statement, objectives, scope, and significance of the study. Lastly, it discussed about the organization of the report.

Chapter 2 focus on the literature review of the study. This chapter starts by the fundamental concepts of finite difference method, followed by presenting some review of previous research done using ADI method.

The research methodology is presented in Chapter 3. The techniques used are ADI method. In this chapter, the algorithm of this method is discussed to solve the groundwater flow equation in two-dimension.

The implementation of ADI method is presented in Chapter 4, where the results are discussed in detail. Finally, Chapter 5 presents the summary, conclusions and recommendations for future study.

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