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Integral Equation Approach for Computing Green's Function on Doubly Connected Regions via the Generalized Neumann Kernel

Siti Zulaiha Aspon[a*], Ali Hassan Mohamed Murid[a,b], Mohamed M. S. Nasser[c], Hamisan Rahmat[a]

[a]Department of Mathematical Sciences, Faculty of Science, UTM, 81310 UTM Johor Bahru, Johor, Malaysia
[b]UTM Centre for Industrial and Applied Mathematics (UTM-CIAM), 81310 UTM Johor Bahru, Johor, Malaysia
[c]Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia

*Corresponding author: sitizulaiha.aspon@yahoo.com

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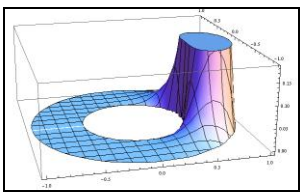
Graphical abstract



Abstract

This research is about computing the Green's function on doubly connected regions by using the method of boundary integral equation. The method depends on solving a Dirichlet problem. The Dirichlet problem is then solved using a uniquely solvable Fredholm integral equation on the boundary of the region. The kernel of this integral equation is the generalized Neumann kernel. The method for solving this integral equation is by using the Nyström method with trapezoidal rule to discretize it to a linear system. The linear system is then solved by the Gauss elimination method. Mathematica plots of Green's functions for several test regions are also presented.

Keywords Green's Function; Dirichlet Problem; Integral Equation; Generalized Neumann Kernel

Abstrak

Kajian ini berkaitan dengan pengiraan fungsi Green pada rantau berkait ganda dua terbatas dengan menggunakan kaedah persamaan kamiran sempadan. Kaedah ini bergantung kepada penyelesaian masalah Dirichlet. Masalah Dirichlet kemudiannya diselesaikan menggunakan persamaan kamiran Fredholm berpenyelesaian unik pada sempadan rantau ini. Inti persamaan kamiran ini adalah inti Neumann teritlak. Kaedah untuk menyelesaikan persamaan kamiran ini ialah dengan menggunakan kaedah Nyström dengan peraturan trapezoid untuk menghasilkan sebuah sistem linear. Sistem linear kemudian diselesaikan dengan kaedah penghapusan Gauss. Plot Mathematica bagi fungsi Green untuk beberapa rantau ujian juga dipersembahkan.

Kata kunci: Fungsi Green; Masalah Dirichlet; Persamaan Kamiran; Inti Neumann Teritlak

1.0 INTRODUCTION

Green's functions are important since they provide a powerful tool in solving differential equations. They are very useful in several fields such as solid mechanics, applied physics, applied mathematics, mechanical engineering, materials science and quantum field theory.[1]

Henrici shows three different methods for computing Green's function for doubly connected regions which leads to three different analytical representations for the Green's function. The methods are the Fourier series method, infinite product method and Theta series method.[2] Crowdy and Marshall have presented an analytical formula for Green's function for Laplace's equation in multiply circular domains. The method is constructive and depends on Schottky-Klein prime function associated with multiply connected circular domain.[3]

Wegmann and Nasser have studied Fredholm integral equation associated with the linear Riemann-Hilbert problems on multiply connected regions with smooth boundary curves. The kernel of these integral equations is the generalized Neumann kernel. They investigated the existence and uniqueness of solutions of the integral equations by determining the exact number of linear independent solutions and their adjoints.[4] Based on Wegmann and Nasser, Nasser *et al*. have proposed a new boundary integral method for the solution of Laplace's equation on multiply connected regions using either Dirichlet boundary condition or the Neumann boundary condition. The method is based on two uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel.[5]

Recently, Alagele has proposed a new method for computing the Green's function on simply connected region by using the method of boundary integral equation which depends on the solution of a Dirichlet problem.[6]

Based on paper by Wegmann and Nasser and Nasser *et al*., the Dirichlet problem is solved using a uniquely solvable Fredholm integral on the equation boundary of the region.[4,5] This paper wish to extend Alagele's work to compute Green's function for bounded doubly connected regions using integral equation with generalized Neumann kernel.[6]

2.0 AUXILIARY MATERIALS

Let Ω be a bounded doubly connected region in the complex plane (Figure 1). The outer boundary Γ_0 has a counter clockwise direction and surrounds the boundary Γ_1 which has clockwise orientation. So we have $\Gamma = \Gamma_0 \cup \Gamma_1$.

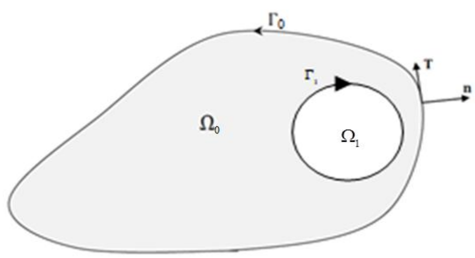


Figure 1 Bounded doubly connected region

We assume that each boundary Γ_k has a parameterization $\eta_k(t),\ t \in J_k,\ k = 0,1,$ which is a complex periodic function with period 2π, where $J_k = [0, 2\pi]$ is the parametric interval for each η_k. The parameterization $\eta_k(t)$ also need to be twice continuously differentiable such that

$$\dot{\eta}_k(t) = \frac{d\eta(t)}{dt} \neq 0. \tag{2.1}$$

Therefore the parameterization η_k of the whole boundary Γ can be written as

$$\begin{aligned} \Gamma_0 &: \eta_0(t),\ \ t \in J_0 = [0, 2\pi], \\ \Gamma_1 &: \eta_1(t),\ \ t \in J_1 = [0, 2\pi]. \end{aligned} \tag{2.2}$$

Let u be a real function defined in the region Ω and let $z = x + iy \in \Omega$. In our research, for simplicity, we write $u(z)$ instead of $u(x,y)$. Let H^α be the space of all real Hölder continuous function with exponent α on the boundary Γ. The interior Dirichlet problem is defined as follows:

Interior Dirichlet problem:

Let $\gamma_k \in H^\alpha$ be a given function. Find the function u harmonic in Ω, Holder continuous on Γ and satisfies the boundary condition

$$u(\eta_k) = \gamma_k, \quad \eta_k \in \Gamma_k, \tag{2.3}$$

where

$$\begin{cases} \gamma_0(t), & t \in J_0, \\ \gamma_1(t), & t \in J_1. \end{cases}$$

The interior Dirichlet problem (2.3) is uniquely solvable and can be regarded as a real part of an analytic function F in Ω which is not necessary single-valued.[2,5] The function F can be written as

$$F(z) = f(z) - a_1 \ln\left(z - z_1\right), \tag{2.4}$$

where f is a single-valued analytic function in Ω, z_1 is a fixed point in Ω_1 and a_1 is real constant uniquely determined by γ_k.[5] We assume for bounded Ω that $\operatorname{Im} f(\alpha) = 0.$ The constant a_1 is chosen to ensure that

$$\int_{\Gamma_1} f'(\eta_k) d\eta = 0.$$

In general the Green's function for Ω can be expressed by[8]

$$G(z, z_0) = u(z) - \frac{1}{2\pi} \ln\left|z - z_0\right|, \quad z, z_0 \in \Omega, \tag{2.5}$$

where u is the unique solution of the interior Dirichlet problem

$$\begin{cases} \nabla^2 u(z) = 0, z \in \Omega \\ u(\eta_k) = \dfrac{1}{2\pi} \ln\left|\eta_k - z_0\right|, \eta_k \in \Gamma_k. \end{cases} \tag{2.6}$$

By computing F given by (2.4) with

$$\gamma_0 = \frac{1}{2\pi} \ln\left|\eta_0 - z_0\right| \text{ and } \gamma_1 = \ln\left|\eta_0 - z_1\right|, \eta_k \in \Gamma_k, \tag{2.7}$$

the unique solution of the interior Dirichlet problem (2.6) is given in $\Omega \cup \Gamma$ by

$$u(z) = \operatorname{Re} F(z). \tag{2.8}$$

3.0 INTEGRAL EQUATION FOR THE INTERIOR DIRICHLET PROBLEM

Let $A_k(t)$ be continuously differentiable 2π-periodic functions for all $t \in J_k$, $k = 0,1$. We consider two real functions[9]

$$N_{l,k}(s,t) = \frac{1}{\pi} \operatorname{Im}\left(\frac{A_l(s)}{A_k(t)} \frac{\dot{\eta}_k(t)}{\eta_k(t) - \eta_l(s)} \right), \tag{3.1}$$

$$M_{l,k}(s,t)=\frac{1}{\pi}\mathrm{Re}\left(\frac{A_l(s)}{A_k(t)}\frac{\dot{\eta}_k(t)}{\eta_k(t)-\eta_l(s)}\right). \qquad (3.2)$$

The kernel $N_{l,k}(s,t)$ is called the *generalized Neumann kernel* formed with complex-valued function $A_k(t)$ and $\eta_k(t)$.[4] When $A_k=1,$ the kernel $N_{l,k}$ is the classical Neumann kernel which arise frequently in the integral equations for potential theory and conformal mapping.[2]

Theorem 3.1[4]

a) *The kernel* $N_{l,k}(s,t)$ *is continuous which takes on the diagonal the values*

$$N_{k,k}(t,t)=\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{2}\frac{\ddot{\eta}_k(t)}{\dot{\eta}_k(t)}-\frac{\dot{A}_k(t)}{A_k(t)}\right). \qquad (3.3)$$

b) *The kernel* $M_{l,k}(s,t)$ *is continuous for* $l\neq k.$ *When* $l=k,$ *the kernel* $M_{k,k}(s,t)$ *has the representation*

$$M_{k,k}(s,t)=-\frac{1}{2\pi}\cot\frac{s-t}{2}+M_{1,k}(s,t), \qquad (3.4)$$

with a continuous kernel $M_{1,k}$ *which takes on the diagonal the values*

$$M_{1,k}(t,t)=\frac{1}{\pi}\mathrm{Re}\left(\frac{1}{2}\frac{\ddot{\eta}_k(t)}{\dot{\eta}_k(t)}-\frac{\dot{A}_k(t)}{A_k(t)}\right), \qquad (3.5)$$

where $k=0,1.$

To find the function $F(z)$ given by (2.4), we need to find the function $f(z)$ and the real constant a_1. We define real functions

$$\gamma_k^{[0]}=\gamma_k \text{ and } \gamma_k^{[1]}=\ln\left|\eta_k-z_1\right| \text{ for } k=0,1, \qquad (3.6)$$

where γ_k satisfy (2.7). It follows that[5]

$$f^{[p]}(\eta_k(t))=\gamma_k^{[p]}(t)+\overset{\circ}{h}{}_k^{[p]}(t)+i\mu_k^{[p]}(t), \text{for } p,\ k=0,1, \qquad (3.7)$$

are boundary values of analytic function $f^{[p]}$ in Ω where $\mu_k^{[p]}$ is the unique solution of the integral equation

$$\mu_l^{[p]}(s)-\int_0^{2\pi}N_{l,0}(s,t)\mu_0^{[p]}(t)dt-\int_0^{2\pi}N_{l,1}(s,t)\mu_1^{[p]}(t)dt=$$
$$-\int_0^{2\pi}M_{l,0}(s,t)\gamma_0^{[p]}(t)dt-\int_0^{2\pi}M_{l,1}(s,t)\gamma_1^{[p]}(t)dt,\quad s\in J_l,\ l,p=0,1. \qquad (3.8)$$

and

$$h_l^{[p]}=\frac{1}{2}\left[\sum_{k=0}^{1}\int_0^{2\pi}M_{l,k}(s,t)\mu_k^{[p]}(t)dt-\gamma_l+\sum_{k=0}^{1}\int_0^{2\pi}N_{l,k}(s,t)\gamma_k^{[p]}(t)dt\right], \qquad (3.9)$$

with

$$\overset{\circ}{h}{}_0^{[p]}=0,$$
$$\overset{\circ}{h}{}_1^{[p]}=h_1^{[p]}-h_0^{[p]}.$$

It follows from Nasser *et al*.[5] that the unknown constant a_1 is the solution of the equation

$$\overset{\circ}{h}{}^{[0]}+a_1\overset{\circ}{h}{}^{[1]}=0. \qquad (3.10)$$

Then

$$a_1=-\left(\frac{h_1^{[0]}-h_0^{[0]}}{h_1^{[1]}-h_0^{[1]}}\right).$$

Hence, the boundary values of the function f is given by[5]

$$f(\eta_k(t))=\gamma_k(t)+a_1\ln|\eta_k(t)-z_1|+i\mu_k(t), \qquad (3.11)$$

where

$$\mu_k(t)=\mu_k^{[0]}(t)+a_1\mu_k^{[1]}(t).$$

By this result we can compute the interior values of $f(z)$ over the whole region Ω by using the Cauchy integral formula

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(w)}{w-z}dw. \qquad (3.12)$$

We then compute the function $F(z)$ from (2.4) and $u(z)$ from (2.8).

4.0 NUMERICAL IMPLEMENTATION

Denoting the right-hand side of the Equation (3.8) by $\psi_l(s)$, we get

$$\mu_l^{[p]}(s)-\int_0^{2\pi}N_{l,0}(s,t)\mu_0^{[p]}(t)dt-\int_0^{2\pi}N_{l,1}(s,t)\mu_1^{[p]}(t)dt=\psi_l^{[p]}(s). \qquad (4.1)$$

Since the functions A_k and η_k are 2π - periodic, the integrals are discretized by the Nyström method with trapezoidal rule.[7]

Let n be a given integer and define the n equidistant collocation points t_j by

$$t_j = (j-1)\frac{2\pi}{n}, \quad j = 1,2,\ldots,n. \tag{4.2}$$

Then, using the Nyström method for (4.1) we obtain the linear system

$$\mu_l^{[p]}(t_i) - \frac{2\pi}{n}\sum_{j=1}^{n} N_{l,0}(t_i,t_j)\mu_0^{[p]}(t_j) - \frac{2\pi}{n}\sum_{j=1}^{n} N_{l,1}(s_i,t_j)\mu_1^{[p]}(t_j) = \psi_l^{[p]}(s_i) \tag{4.3}$$

where μ_k is an approximation to μ, and

$$N_{l,k}(t_i,t_j) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{A_l(t_i)}{A_k(t_j)}\dfrac{\dot\eta_k(t_j)}{\eta_k(t_j)-\eta_l(t_i)}\right], & l \neq k, \text{ or } l = k, t_i \neq t_j \\ \dfrac{1}{2\pi}\mathrm{Im}\left[\dfrac{\ddot\eta_k(t_j)}{\dot\eta_k(t_j)}\right] - \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{\dot A_k(t_j)}{A_k(t_j)}\right], & l = k, t_i = t_j, \end{cases}$$

and

$$\psi_l^{[p]}(t_i) = \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_i - t}{2}\gamma_k^{[p]}(t_j)dt - \frac{2\pi}{n}\sum_{j=1}^{n} M_{1,k}(t_i,t_j)\gamma_k^{[p]}(t_j)$$

$$-\frac{2\pi}{n}\sum_{j=1}^{n} M_{l,k}(t_i,t_j)\gamma_k^{[p]}(t_j), \tag{4.4}$$

where $l,k,p = 0,1,$ and

$$M_{l,k}(t_i,t_j) = \frac{1}{\pi}\mathrm{Re}\left(\frac{A_l(t_i)}{A_k(t_j)}\frac{\dot\eta_k(t_j)}{\eta_k(t_j)-\eta_l(t_i)}\right), \; l \neq k,$$

$$M_{1,k}(t_i,t_j) = \begin{cases} \dfrac{1}{\pi}\mathrm{Re}\left[\dfrac{A_k(t_i)}{A_k(t_j)}\dfrac{\dot\eta_k(t_j)}{\eta_k(t_j)-\eta_k(t_i)}\right] + \dfrac{1}{2\pi}\cot\dfrac{t_i - t_j}{2}, & t_i \neq t_j \\ \dfrac{1}{\pi}\mathrm{Re}\left[\dfrac{1}{2}\dfrac{\ddot\eta_k(t_j)}{\dot\eta_k(t_j)} - \dfrac{\dot A_k(t_j)}{A_k(t_j)}\right], & t_i = t_j. \end{cases}$$

We use Wittich method to approximate the integral that contains cotangent function and obtain[10]

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_i - t}{2}\gamma_k(t)dt = \sum_{j=1}^{n} K(i,j)\gamma_k(t_j) \tag{4.5}$$

where

$$K(i,j) = \begin{cases} 0, & \text{if } j\text{-}i \text{ is even,} \\ \dfrac{2}{n}\cot\dfrac{(i-j)\pi}{n}, & \text{if } j\text{-}i \text{ is odd.} \end{cases}$$

The left-hand side of (4.3) can also be calculated directly by using *Mathematica*. Define the matrices $P = [P_{ij}],\ Q = [Q_{ij}],\ R = [R_{ij}],\ S = [S_{ij}]$ and vectors $\vec{x}_l = [x_{l,i}]$ and $\vec{y}_l = [y_{l,i}]$ by

$$P_{ij} = \frac{2\pi}{n}N_{l,k}(\eta_0(t_i),\eta_0(t_j)),\ Q_{ij} = \frac{2\pi}{n}N_{l,k}(\eta_0(t_i),\eta_1(t_j)),$$

$$R_{ij} = \frac{2\pi}{n}N_{l,k}(\eta_1(t_i),\eta_0(t_j)),\ S_{ij} = \frac{2\pi}{n}N_{l,k}(\eta_1(t_i),\eta_1(t_j)),$$

$$x_{l,i} = \mu_l(t_i),\ y_{l,i} = \psi_l(t_i).$$

Hence, the Equation (4.3) can be written as an $2n$ by $2n$ system

$$\begin{aligned} (I-P)\vec{x}_0 - Q\vec{x}_1 &= \vec{y}_0, \\ -R\vec{x}_0 + (I-S)\vec{x}_1 &= \vec{y}_1. \end{aligned} \tag{4.6}$$

To solve the system (4.6) we use the method of Gaussian elimination. Since (4.1) has a unique solution, then for a wide class of quadrature formula the system (4.6) also has a unique solution, as long as n is sufficiently large.[7] After we get the unique solution $x_{l,i} = \mu_l(t_i)$, then we calculate $f(\eta_0(t_j))$ and $f(\eta_1(t_j))$ by using the following formula:

$$\begin{aligned} f(\eta_0(t_j)) &= \gamma_0(t_j) + a_1\ln|\eta_0(t_j) - z_1| + i\mu_0(t_j), \\ f(\eta_1(t_j)) &= \gamma_1(t_j) + a_1\ln|\eta_1(t_j) - z_1| + i\mu_1(t_j), \end{aligned} \tag{4.7}$$

which represents the boundary values of $f(z)$ on Γ. We can compute the interior values of $f(z)$ over the whole region Ω by using the Cauchy integral formula given in (3.12), i.e.,

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\eta_0(t_j))}{\eta_0(t_j)-z}\eta_0'(t_j)dt + \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\eta_1(t_j))}{\eta_1(t_j)-z}\eta_1'(t_j)dt \tag{4.8}$$

To increase the accuracy of $f(z)$ we shall use the following formula. Based on the fact that $\frac{1}{2\pi i}\int_\Gamma \frac{1}{\eta - z}d\eta = 1$, we can write $f(z)$ as

$$f(z) = \frac{\dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{f(\eta_0(t_j))}{\eta_0(t_j)-z}\eta_0'(t_j)dt + \dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{f(\eta_1(t_j))}{\eta_1(t_j)-z}\eta_1'(t_j)dt}{\dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{\eta_0'(t_j)}{\eta_0(t_j)-z}dt + \dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{\eta_1'(t_j)}{\eta_1(t_j)-z}dt}. \tag{4.9}$$

Then, using the Nyström method with the trapezoidal rule to discretize the integrals in (4.9), we obtain the approximation

$$f(z) = \frac{\displaystyle\sum_{j=1}^{n}\frac{f(\eta_0(t_j))}{\eta_0(t_j)-z}\eta_0'(t_j) + \sum_{j=1}^{n}\frac{f(\eta_1(t_j))}{\eta_1(t_j)-z}\eta_1'(t_j)}{\displaystyle\sum_{j=1}^{n}\frac{\eta_0'(t_j)}{\eta_0(t_j)-z} + \sum_{j=1}^{n}\frac{\eta_1'(t_j)}{\eta_1(t_j)-z}}. \tag{4.10}$$

This has the advantage that the denominator in this formula compensates for the error in the numerator.[11] Next, substitute $f(z)$ given in (4.10) into Equation $F(z)$ in (2.4), and by taking the real part of (2.4) gives (2.8), i.e.

$$u(z) = \operatorname{Re} F(z).$$

Finally, by using $u(z)$ we can compute the Green's function $G_n(z, z_0)$ by the following formula (2.5), i.e.

$$G_n(z, z_0) = u(z) - \frac{1}{2\pi}\ln|z - z_0|.$$

■5.0 NUMERICAL EXAMPLES

Example 1

In Example 1, we consider an annulus as shown (Figure 2). The boundary of this region is parameterized by the function

$$\begin{aligned}\Gamma_0 &: \eta_0(t) = e^{it}, \\ \Gamma_1 &: \eta_1(t) = pe^{-it},\end{aligned} \qquad 0 \le t \le 2\pi$$

with $p = 0.5,\ z_0 = 0.75,$

$$\gamma_0 = \frac{1}{2\pi}\ln|\eta_0 - z_0| \text{ and } \gamma_1 = \ln|\eta_0 - z_1|.$$

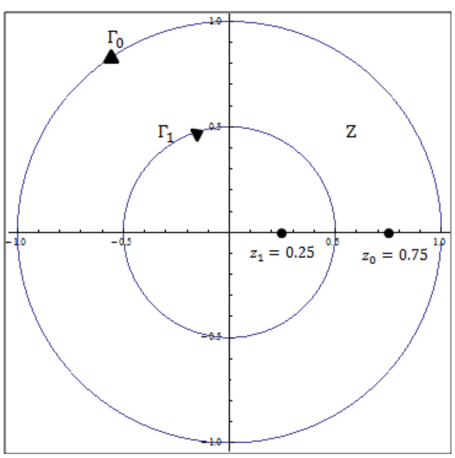


Figure 2 The test region Ω for Example 1

The exact Green's function of this region is given by[2]

$$G(z,z_0) = \left(\frac{\operatorname{Log}\rho}{\operatorname{Log}\mu} - 1\right)\operatorname{Log}\tau - \operatorname{Log}\left|\frac{1-\tau^{-1}\rho e^{i\sigma}}{1-\tau\rho e^{i\sigma}}\right| + \sum_{n=1}^{\infty}\frac{\mu^n}{n}\frac{\tau^n - \tau^{-n}}{\mu^n - \mu^{-n}}\left(\rho^n - \rho^{-n}\right)\cos n\sigma$$

where $z = \rho e^{i\sigma}$, and the infinite series converges, uniformly for $\mu \le |z| \le 1$ and $\mu \le \tau \le 1$ at least like a geometric series with ratio μ.

We describe the error by maximum error norm $\|G(z,z_0) - G_n(z,z_0)\|_\infty$, where n is the number of nodes and $G(z,z_0)$ is the numerical approximation of $G(z,z_0)$. We choose some test points inside the region. The results are shown in Table 1.

Table 1 The error $\|G(z,z_0) - G_n(z,z_0)\|_\infty$

z \ n	**32**	**64**	**128**
0.6	1.72×10^{-4}	1.13×10^{-6}	5.81×10^{-11}
0.7	9.77×10^{-5}	7.86×10^{-7}	4.05×10^{-11}
0.8	1.58×10^{-5}	4.72×10^{-7}	2.54×10^{-11}
0.9	1.88×10^{-4}	2.29×10^{-8}	1.15×10^{-11}
0.74999999999	4.8×10^{-5}	6.3×10^{-7}	3.27×10^{-11}
0.75000000001	4.8×10^{-5}	6.3×10^{-7}	3.27×10^{-11}
0.5+0.5i	1.35×10^{-4}	7.67×10^{-7}	3.94×10^{-11}
0.6+0.6i	7.78×10^{-5}	3.69×10^{-7}	1.87×10^{-11}
0.7+0.7i	1.46×10^{-5}	7.21×10^{-8}	3.03×10^{-12}

The 3D plot of the surface of $G_n(z,z_0)$ is shown (Figure 3).

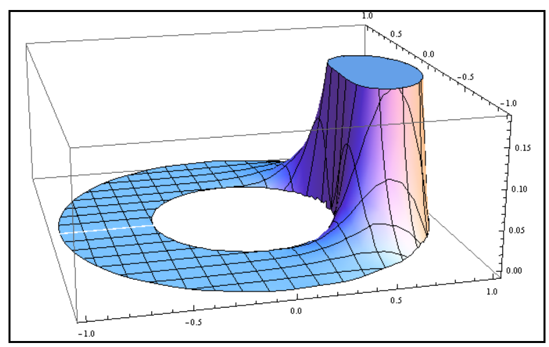


Figure 3 The 3D plot of Green's function for Example 1

Example 2

In Example 2, we consider an Epitrochoid as shown (Figure 4). The boundary of this region is parameterized by the function

$$\begin{aligned}\Gamma_0 &: \eta_0(t) = e^{it} + pe^{-it}, \\ \Gamma_1 &: \eta_1(t) = qe^{-it},\end{aligned} \qquad 0 \le t \le 2\pi$$

with $p = 0.3333,\ q = 0.1,\ z_0 = 0.75,\ z_1 = 0.01,$

$$\gamma_0 = \frac{1}{2\pi}\ln|\eta_0 - z_0| \text{ and } \gamma_1 = \ln|\eta_0 - z_1|.$$

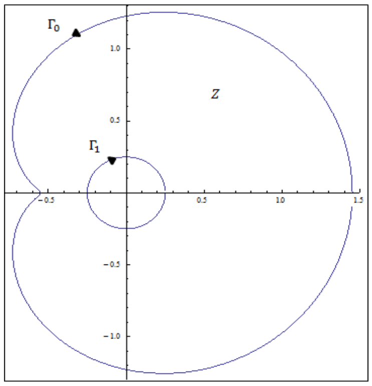


Figure 4 The test region Ω for Example 2

The 3D plot of the surface of $G_n(z,z_0)$ is shown (Figure 5).

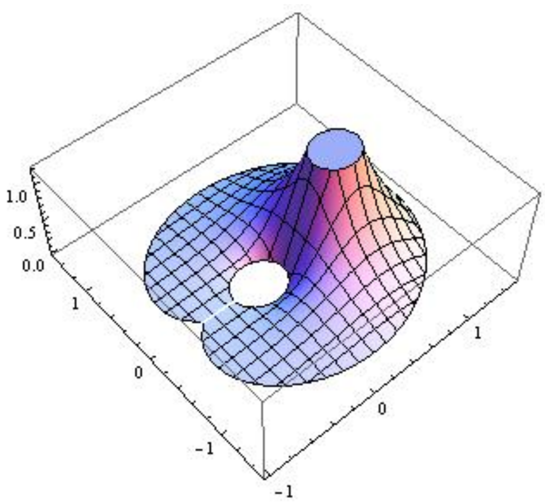


Figure 5 The 3D plot of Green's function for Example 2

6.0 CONCLUSION

This study has presented a method for computing the Green's function on doubly connected regions by using a new approach based on boundary integral equation with generalized Neumann kernel. The idea for computing the Green's function on Ω is to solve the Dirichlet problem

$$\begin{cases} \nabla^2 u(z) = 0, z \in \Omega \\ u(\eta_k) = \dfrac{1}{2\pi} \ln|\eta_k - z_0|, \eta_k \in \Gamma_k. \end{cases} \quad (6.1)$$

on that region by means of solving an integral equation numerically using Nyström method with the trapezoidal rule. Once we got the solution $u(z)$, the Green's function of Ω can be computed by using the formula

$$G_n(z,z_0) = u(z) - \frac{1}{2\pi} \ln|z - z_0|. \quad (6.2)$$

The numerical example illustrates that the proposed method can be used to produce approximations of high accuracy.

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