

# Free vibration of symmetric angle-ply laminated circular cylindrical shells

**K K Viswanathan<sup>1</sup>, Zainal Abdul Aziz, H Z Amirah, Saira Javed**

UTM-Center for Industrial and Applied Mathematics (UTM-CIAM)  
Department of Mathematical Sciences, Faculty of Science,  
Universiti Teknologi Malaysia, 81310 Skudai, Johor Bahru, Johor, Malaysia  
e-mail: visu20@yahoo.com, viswanathan@utm.my

**Abstract.** Free vibration of symmetric angle-ply laminated circular cylindrical shells is studied using Spline approximation. The equations of motions in longitudinal, circumferential and transverse displacement components, are derived using Love's first approximation theory. The coupled differential equations are solved using Spline approximation to obtain the generalized eigenvalue problem. Parametric studies are performed to analyse the frequency response of the shell with reference to the material properties, number of layers, ply orientation, length and circumferential node number and different boundary conditions.

## 1. Introduction

Shell structures are widely used as structural components in engineering, aerospace, naval and chemical industry. They are often used as load-bearing structures for aircrafts, rockets, submarines and missile bodies. The cylindrical shells used in those designs are stiffened to achieve better strength, stiffness, temperature resistant and light weight characteristics. Recently, '[1]' studied laminated shells. Free vibration of layered cylindrical shells of variable thickness was examined and the frequencies were analyzed for specially orthotropic materials using collocation with splines '[2]'. The paper on angle-ply laminated cylindrical shells based on the Love-type version of a unified shear deformable shell theory was studied by '[3]'. A refined high-order global-local theory was used to analyse the laminated composite beams using the finite element method including shear deformation '[4]'. Recently, Topal and Uzman '[5]' analyzed thermal buckling load optimization of angle-ply laminated cylindrical shells. Recently, '[6]' investigated the free vibration of symmetric angle-ply cylindrical shells of variable thickness.

Most of the researchers analyzed the problems of free vibration for angle-ply laminated shells without using spline approximation. Thus, there may not be available sufficient work on angle-ply laminated cylindrical shells with orthotropic materials using spline approximation technique in the past literature. This paper studies the free vibrations of angle-ply layered circular cylindrical shells using spline approximation where the problem is formulated by extending Love's first approximation theory on homogenous shell. The shell is made up of uniform layers of isotropic or specially orthotropic materials.

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<sup>1</sup> Corresponding author: E-mail: [visu20@yahoo.com](mailto:visu20@yahoo.com), [viswanathan@utm.my](mailto:viswanathan@utm.my)



Studies are carried out for cylindrical shells with clamped-clamped and simply supported boundary conditions along the axial direction and the layers of the material are considered to be thin. The layers are perfectly bounded together to move without interface slip. The effects of the angle-ply, different materials and geometric parameters on the frequencies of angle-ply laminated cylindrical shells under different boundary conditions are analysed. In the analysis, spline approximation technique will be adopted to approximate the displacement functions to analyse the frequencies.

## 2. Formulation of the Problem

Consider a thin laminated circular cylindrical shell of having the radius  $r$ , length  $\ell$  and thickness  $h$ . Each layer in the laminated composite is assumed to be homogeneous, linearly elastic, and specially orthotropic. The layers are perfectly bonded at the interfaces. The coordinate system  $(x, \theta, z)$  is defined at the mid surface of the shell and  $u, v$  and  $w$  are the displacements in the directions of  $x, \theta$  and  $z$  respectively.

The equations of motion for thin circular cylindrical shells are given by the following (Viswanathan *et al.*, 2010):

$$\frac{\partial N_x}{\partial x} + \frac{1}{r} \frac{\partial N_{x\theta}}{\partial \theta} = \rho h \frac{\partial^2 u}{\partial t^2} \quad (1)$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r} \frac{\partial N_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial M_{x\theta}}{\partial x} + \frac{1}{r^2} \frac{\partial M_\theta}{\partial \theta} = \rho h \frac{\partial^2 v}{\partial t^2} \quad (2)$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2}{r} \frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{r^2} \frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{1}{r} N_{\theta\theta} = \rho h \frac{\partial^2 w}{\partial t^2} \quad (3)$$

The stress resultants and stress couples are given by

$$(N_x, N_\theta, N_{x\theta}) = \int_z (\sigma_x, \sigma_\theta, \sigma_{x\theta}) dz, (M_x, M_\theta, M_{x\theta}) = \int_z (\sigma_x, \sigma_\theta, \sigma_{x\theta}) z dz \quad (4)$$

where  $N_x, N_\theta, N_{x\theta}$  are stress resultants,  $M_x, M_\theta, M_{x\theta}$  are moment resultants and  $\sigma_x, \sigma_\theta, \sigma_{x\theta}$  are stresses in respective directions.

For a thin laminated cylindrical shell, the stress and strain relation of the  $k$ th layer is defined as

$$\{\sigma^{(k)}\} = [Q^{(k)}] [\varepsilon^{(k)}], \quad (5)$$

where  $Q^{(k)}$  are elastic coefficients and  $\varepsilon^{(k)}$  are normal and shear strains of the  $k$ -th layer.

When the materials are oriented at an angle  $\theta$  with the  $x$ -axis, the transformed stress-strain relations are given by

$$\{\sigma^{(k)}\} = [\bar{Q}^{(k)}] [\varepsilon^{(k)}], \quad (6)$$

where  $[\bar{Q}^{(k)}] = [T]^{-1} [Q^{(k)}] [T]$  is defined in '[7]'.

Substituting (6) into (4), we get the equations of stress resultants and moment resultants as follows.

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa \end{bmatrix} \quad (7)$$

where  $\varepsilon_x = \partial u / \partial x$ ,  $\varepsilon_\theta = (1/r) \partial v / \partial \theta + w/r$ ,  $\varepsilon_{x\theta} = \partial v / \partial x + (1/r) \partial u / \partial \theta$ ,  $\kappa_{xx} = -\partial^2 w / \partial x^2$ ,

$\kappa_{\theta\theta} = (1/r^2) \partial v / \partial \theta - (1/r^2) \partial^2 w / \partial \theta^2$  and  $\kappa_{x\theta} = (1/r) \partial v / \partial x - (2/r) \partial^2 w / \partial x \partial \theta$  (8)

Here  $u$ ,  $v$  and  $w$  are the displacement functions of the mid plane in longitudinal, circumferential and transverse directions and  $A_{ij}$ ,  $B_{ij}$  and  $D_{ij}$  are the laminate stiffnesses defined by

$$A_{ij} = \sum \bar{Q}_{ij}^{(k)} (z_k - z_{k-1}), B_{ij} = \sum \bar{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2), D_{ij} = \sum \bar{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3), i, j = 1, 2, 6. \quad (9)$$

Here  $z_k$  is the distance of the  $k$ -th layer from the reference surface.

It is assumed that there are no stretching-shearing, twisting-shearing and symmetric angle-ply lamination, so that  $A_{16} = A_{26} = D_{16} = D_{26} = 0$  and all  $B_{ij} = 0$ . Substituting 'equation (7)' in to the 'equations (1-3)' one can obtain the differential equations in terms of displacement functions  $u, v$  and  $w$ .

The displacement components  $u, v$  and  $w$  are assumed in the separable form given by

$$u(x, \theta, t) = U(x) \cos n\theta e^{i\omega t}, v(x, \theta, t) = V(x) \sin n\theta e^{i\omega t} \text{ and } w(x, \theta, t) = W(x) \cos n\theta e^{i\omega t} \quad (10)$$

where  $x$  and  $\theta$  are the coordinates defined in the longitudinal and circumferential directions respectively,  $\omega$  is the angular frequency of vibration,  $t$  is the time and  $n$  is the circumferential node number.

Substituting the 'equation (10)' in the coupled differential equations, one can obtain the equations in terms  $U$ ,  $V$  and  $W$  and can be written in the matrix form as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = [0] \quad (11)$$

where  $L_{ij}$  are linear differential operators in  $X$ .

The differential equations on the displacement function contain derivatives of third order in  $U$ , second order in  $V$  and forth order in  $W$ . As such they are not amenable to the solution procedure. Hence the equations are combined within themselves and a modified set of equations are derived. The procedure adopted to this end is to differentiate the first of 'equation (11)' with respect to  $x$  once and to use it to eliminate  $U'''$  in the third equation. The modified set of equations are of order 2 in  $U$ , order 2 in  $V$  and order 4 in  $W$ .

### 2.1. Non-Dimensionlization

The following non dimensional parameters are introduced:

$$\begin{aligned} \lambda &= \ell \omega (R_0 / A_{11})^{1/2}, \text{ a frequency parameter;} \\ \delta_k &= h_k / h, \text{ the relative thickness of the } k\text{-th layer;} \\ H &= h / r, \text{ the thickness parameter;} L = \ell / r, \text{ a length parameter;} \\ X &= x / \ell, \text{ a distance co-ordinate;} R = r / \ell, \text{ a radius parameter} \end{aligned} \quad (12)$$

where  $\ell$  is the length of the cylinder,  $r$  is its radius,  $h_k$  is the thickness of the  $k$ -th layer,  $h$  is the total thickness of the shell,  $R_0$  is the inertial coefficient and  $A_{11}$  is a standard extensional rigidity coefficient. Clearly the range of  $X$  is lies between 0 and 1.

### 2.2. Method of Solution

The displacement functions  $U(X)$ ,  $V(X)$  and  $W(X)$  are approximated by the cubic and quantic splines

as stated below:

$$U(X) = \sum_{i=0}^2 a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j), V(X) = \sum_{i=0}^2 c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 H(X - X_j) \quad \text{and} \\ W(X) = \sum_{i=0}^4 e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 H(X - X_j) \quad (13)$$

Here  $H(X-X_j)$  is the Heaviside step function and  $N$  is the number of intervals into which the range  $[0,1]$  of  $X$  is divided. The points  $X=X_s=s/N$  ( $s=0,1,2,\dots,N$ ) are chosen as the knots of the splines, as well as the collocation points. By applying the collocation points to the differential equations given in (11), a set of  $3N+3$  homogeneous equations in  $3N+11$  unknown spline coefficients  $a_i, b_j, c_i, e_k, f_j$  ( $i=0,1,2; k=0,1,2,3,4; j=0,1,2, \dots, N-1$ ), are obtained. To get 8 more equations, we consider the boundary conditions as follows:

(i) (C - C): both the ends clamped and (ii) (S - S): both the end are simply supported.

Combining any one of the boundary conditions, resulting a generalized eigenvalue problem as follows:

$$[M] \{q\} = \lambda^2 [P] \{q\} \quad (14)$$

where  $[M]$  and  $[P]$  are matrices of order  $(3N+7) \times (3N+7)$ ,  $\{q\}$  is a column matrix of order  $(3N+7) \times 1$  and  $\lambda$  is the eigenparameter.

## 2. Results and Discussions

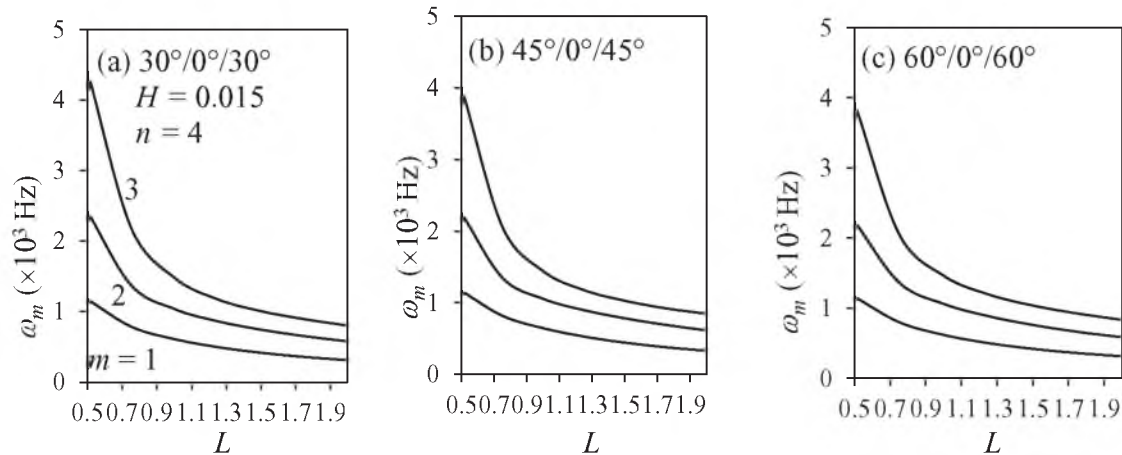
‘Table 1’ depicts comparative study made for homogeneous cylindrical shells of circumferential node number  $n$  and the natural frequencies obtained for the shell with C–C boundary conditions and compared with the results with the values of the natural frequencies obtained by ‘[8-12]’ for the axial mode values of  $m=1,2,3$  and the circumferential mode values for  $n=2,4,6$  are compared with the corresponding results obtained by the present method. The parametric values used are  $\ell=511.2\text{mm}$ ,  $r=216.2\text{mm}$ ,  $h=1.5\text{mm}$ ,  $E=1.83 \times 10^{11} \text{ N/m}^2$ ,  $\nu=0.3$ ,  $\rho=7492\text{kg/m}^3$  ‘[13]’. The agreement of the current results are very good. The slightly higher differences with the other results quoted must be due to the differences in the formulations, methods of solution and numerical accuracy involved.

**Table 1.** Comparison of Natural Frequencies for Cylinder Shells with Clamped Edges

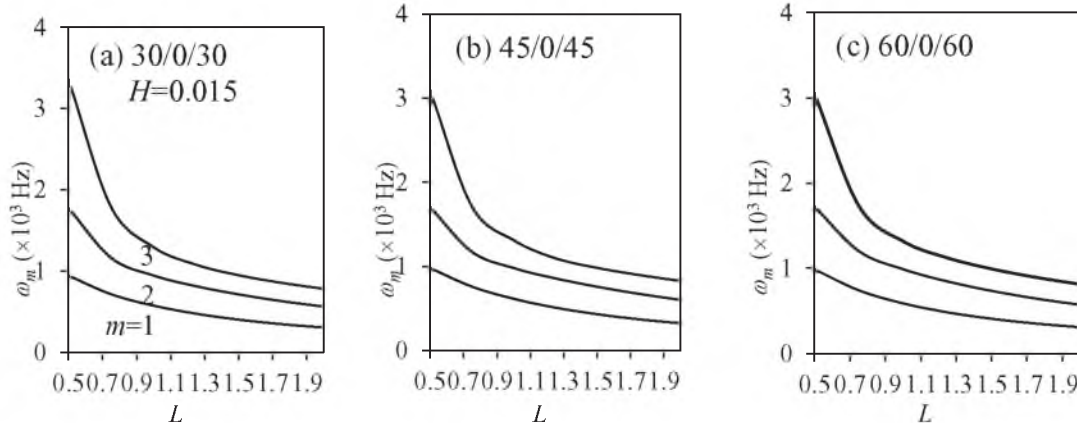
Mode		Natural frequencies in Hz					
Axial mode $m$	Circumferential node $n$	Present Method	Viswanathan and Navaneetha-krishnan ‘[8]’	Arnold and Warburton [12]	Smith and Haft [11]	Au-Yang [10]	Goncalves and Ramos [9]
1	2	1312	1299	1240	1429	1485	1405
1	4	3920	3872	3970	4142	4147	4004
1	6	9195	9125	9230	9400	9396	9251
2	2	2530	2660	2440	2682	2852	2904
2	4	4123	4043	4160	4335	4350	4214
2	6	9283	9276	9380	9840	9533	9384

‘Figure 2’ describes the effect of length parameter  $L$  with the angular frequencies  $\omega_m \times 10^3 \text{ Hz}$  ( $m=1,2,3$ ) for three layered shells having ply orientation of  $30^\circ/0^\circ/30^\circ$ ,  $40^\circ/0^\circ/45^\circ$  and  $60^\circ/0^\circ/60^\circ$  and with arranging materials in the order of S-glass epoxy (SGE) – High strength graphite epoxy (HSG)- S-glass epoxy (SGE) under C-C boundary conditions fixing thickness  $H=0.015$  and circumferential node number  $n=4$ . It is observed from the figure, that  $\omega_m$  decreases with  $L$  increasing. The decrease is fast

for very short shells ( $0.5 < L < 0.75$ ). Here  $\omega_m$  decreases rapidly for the value of  $L$  between 0.75 and 0.85 after which the decrement is very low for  $L > 0.85$ . The nature of variation is same for all the modes and the values are higher for higher modes. The per cent changes in  $\omega_m$  ( $m=1,2,3$ ) over the range  $0.5 < L < 2$  are 276.66%, 317.79%, 449.57% for  $30^\circ/0^\circ/30^\circ$ , 254.37%, 266.61%, 373.70% for  $45^\circ/0^\circ/45^\circ$ , and 269.74%, 282.63%, 371.51% for  $60^\circ/0^\circ/60^\circ$  respectively.



**Figure 2.** Effect of length of three layered shell on angular frequencies for  $n = 4$  under  $C - C$  boundary conditions.

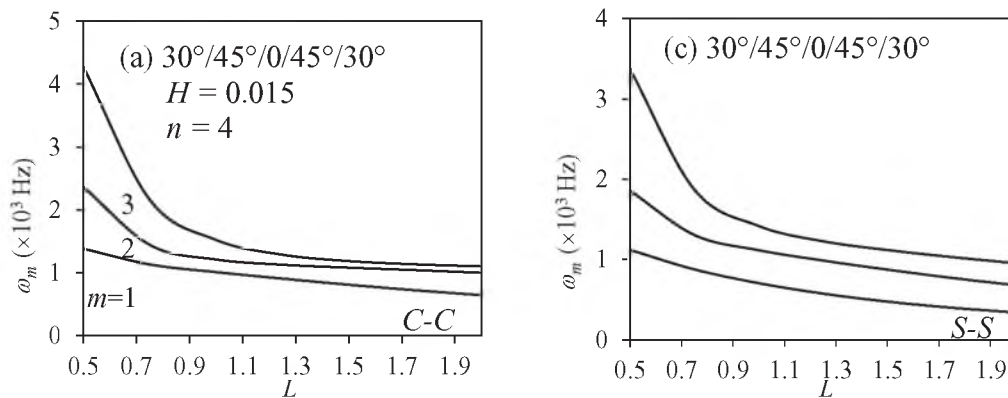


**Figure 3.** Effect of the length of three layered cylindrical shells with angular frequencies under  $S - S$  boundary conditions.

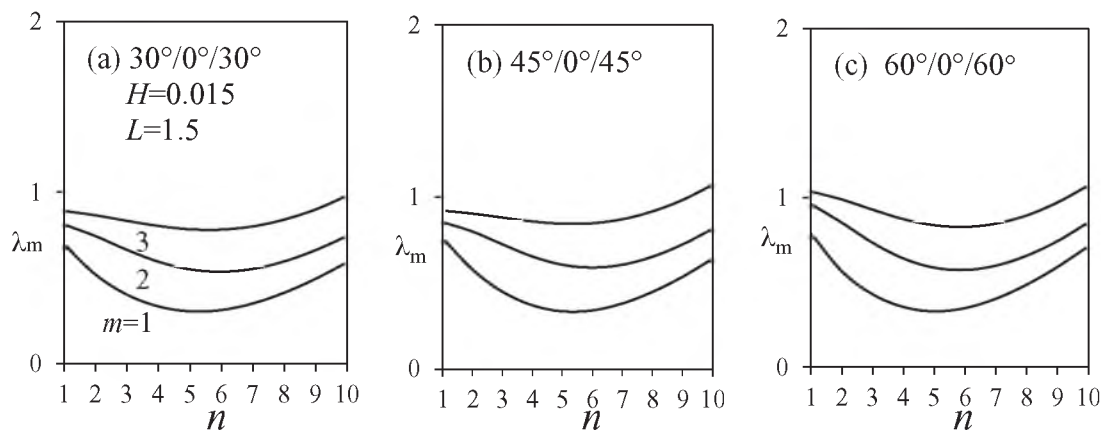
The effect of length parameter  $L$  on the angular frequency parameter  $\omega_m \times 10^3 \text{ Hz}$  ( $m=1,2,3$ ) under  $S - S$  boundary conditions for three layered shells with ply orientation of  $30^\circ/0^\circ/30^\circ$ ,  $45^\circ/0^\circ/45^\circ$  and  $60^\circ/0^\circ/60^\circ$  with the order of the materials in the order of SGE-HSG-SGE are shown in ‘figure 3’.

‘Figure 4’ describes the variation of angular frequencies  $\omega_m \times 10^3 \text{ Hz}$  ( $m=1,2,3$ ) with respect to length parameter  $L$  for five layered plates arranging the materials in the order of SGE-HSG-SGE-HSG-SGE with ply angles of  $30^\circ/45^\circ/0^\circ/45^\circ/30^\circ$  under  $C-C$  and  $S-S$  boundary conditions by fixing  $H=0.015$  and  $n=4$ . As

$L$  increases,  $\omega_m$  is observed to be decreased in general. The decrease is fast for very short shells ( $0.5 < L < 0.75$ ). There is a rapid change in the rate of decrease of  $\omega_m$  in the  $0.75 < L < 0.85$  after which the decrease is very low. The value of angular frequency  $\omega_m$  is higher for higher modes. The per cent changes in  $\omega_m$  ( $m=1,2,3$ ) over the range  $0.5 < L < 2$  for  $30^\circ/45^\circ/0/45^\circ/30^\circ$  ply orientation under  $C-C$  boundary conditions are 253.24%, 276.82%, 349.83% and 174.11%, 230.82%, 254.62% for  $S-S$  boundary conditions.



**Figure 4.** Effect of the length of five layered cylindrical shells with angular frequencies under  $C-C$  and  $S-S$  boundary conditions.



**Figure 5.** Variation of frequency parameter with respect to the circumferential node number for three layered shells under  $C-C$  boundary conditions.

‘Figure 5’ shows the influence of the circumferential node number  $n$  of the frequency parameter  $\lambda_m$  ( $m=1,2,3$ ) for three layered shells with ply orientation of  $30^\circ/0^\circ/30^\circ$ ,  $45^\circ/0^\circ/45^\circ$  and  $60^\circ/0^\circ/60^\circ$  with



materials in the order of SGE-HSG-SGE fixing thickness  $H=0.015$  and length  $L=1.5$  under  $C-C$  boundary conditions. The per cent changed in  $\lambda_l$  for three cases considered are (a) 9.04%, (b) 3.11% and (c) 5.53%. It is seen at the outset that  $n$  affect  $\lambda_m$  significantly. The effect is higher for lower meridional mode values. In general,  $\lambda_m$  decreases with  $n$  up to some value of  $n$  and then increases which occurs approximately in the range  $4 < n < 6$ .

### 3. Conclusions

Layering of the shell wall influences the natural frequencies of the vibration of the cylindrical shell. The relative thickness, as well as the angle ply orientation between them affects the frequencies. A desired frequency of vibration, within a range of frequencies, may be obtained by a proper choice of the relative thickness of layers and the angle-ply rotation among the chosen materials of the layers. We can choose the desired frequency of vibration from the results by a proper choice of the coefficient of thickness variations and arrangements of ply-angles.

### 4. Acknowledgement

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