Approximate Analytical Solutions of KdV and Burgers' Equations via HAM and nHAM

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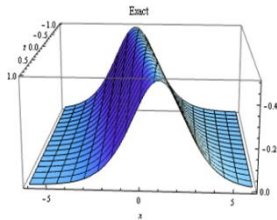
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Graphical abstract

Abstract

This article presents a comparative study of the accuracy between homotopy analysis method (HAM) and a new technique of homotopy analysis method (nHAM) for the Korteweg–de Vries (KdV) and Burgers' equations. The resulted HAM and nHAM solutions at 8th-order and 6th-order approximations are then compared with that of the exact soliton solutions of KdV and Burgers' equations, respectively. These results are shown to be in excellent agreement with the exact soliton solution. However, the result of HAM solution is ratified to be more accurate than the nHAM solution, which conforms to the existing finding.

Keywords: KdV equation; Burgers' equation; homotopy analysis method; new homotopy analysis method; approximate analytic solution

Abstrak

Artikel ini mempersembahkan satu kajian perbandingan ketepatan di antara kaedah analisis homotopi (HAM) dan suatu teknik baru kaedah analisis homotopi (nHAM) bagi menyelesaikan persamaan Korteweg-de Vries (KdV) dan Burgers'.Hasil penyelesaian HAM dan nHAM pada peringkat anggaran kelapan dan keenam itu dibandingkan dengan penyelesaian tepat bagi persamaan KdVdan Burgers masing-masing. Keputusan ini didapati dalam persetujuan cemerlang dengan penyelesaian tepat soliton. Walaubagaimanapun, keputusan bagi penyelesaian HAM ditunjukkan lebih tepat daripada penyelesaiann HAM, yang mematuhi hasil penemuan terkini.

Kata kunci: Persamaan KdV; persamaan Burgers'; kaedah analisis homotopi; kaedah analisis homotopi baru; penyelesaian analisis beranggaran

■1.0 INTRODUCTION

Korteweg and de Vries derived KdV equation to model Russell's observation of the phenomenon of solitons in 1895. The Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modelling of gas dynamics and traffic flow. The first steady-state solution of Burgers' equation was given by Bateman in 1915. There are several analytical techniques of studying the integrable nonlinear wave equations that have soliton solutions [1-3]. It is customarily difficult to solve nonlinear problems, especially by analytic technique. Therefore, seeking suitable solving methods (i.e. exact, approximate or numerical) is an active task in branches of engineering mathematics. Recently, a new approximate analytic approach named homotopy analysis method (HAM) has seen rapid development. The homotopy analysis method (HAM) [4, 5] is a powerful analytic technique for solving non-linear problems, which was initially introduced by Liao in 1992. Recently, this technique has been effectively applied to several non-linear problems in science and engineering, such as generalized Hirota-Satsuma coupled KdV equation [6], third grade fluid past a porous plate [7], non-linear flows with slip boundary condition [8], the KdV and Burgers' equations [9] and more recently Aziz *et al.* [10] examined constant accelerated flow for a third-grade fluid in a porous medium and a rotating frame. In this direction, the effectiveness, flexibility and validity of HAM are confirmed through all of these successful applications. In addition, several different kinds of non-linear problems were solved via HAM (see Liao [11-13], Abbasbandy [14, 15] and Abbasbandy & Shirzadi [16]). More recently, a powerful

modification of HAM was proposed in [17-19]. This modification only deals with the non-homogeneous term or variable coefficients through their series expansion. Nonetheless, Hassan and El-Tawil in [20, 21] have successfully applied a new technique of HAM (or nHAM for short) to obtain an approximation of some high-order in *t*-derivative (order $n \geq 2$) nonlinear partial differential equations. We are concerned with the accuracy between Liao's HAM and Hassan and El-Tawil's nHAM in solving the KdV and Burgers' equations and their conclusions in [20, 21].

The paper layout is as follows: In Section 2, basic idea of HAM is presented. In Section 3, basic idea of nHAM is given. In Section 4, the KdV equation is approximately solved by HAM method. In Section 5, the KdV equation is approximately solved by nHAM method. In Sections 6 and 7, the Burgers' equation is approximately solved by HAM and nHAM methods respectively. In Section 8, HAM and nHAM results are compared and discussed. Section 9 provides a brief conclusion.

■2.0 IDEAS OF HAM

Consider a nonlinear equation in a general form,

$$\mathcal{N}\left[u\left(r,t\right)\right]=0, \tag{1}$$

where $\mathcal{N}$ indicates a nonlinear operator, $u\left(r,t\right)$ an unknown function. Suppose $u_0\left(r,t\right)$ denotes an initial guess of the exact solution $u\left(r,t\right)$, $\mathcal{H}\left(r,t\right) \neq 0$ an auxiliary function, ℓ an auxiliary linear operator, $\hbar \neq 0$ an auxiliary parameter, and $q \in \left[0,1\right]$ as an embedding parameter. By means of HAM, we construct the so-called zeroth-order deformation equation

$$\left(1-q\right)\ell\left[\phi\left(r,t;q\right)-u_0\left(r,t\right)\right]=$$
$$q\hbar\mathcal{H}\left(r,t\right)\mathcal{N}\left[\phi\left(r,t;q\right)\right]. \tag{2}$$

It should be noted, that the auxiliary parameter attributes in HAM are chosen with freedom. Obviously, when $q=0,1$ it holds

$$\phi\left(r,t;0\right)=u_0\left(r,t\right), \ \phi\left(r,t;1\right)=u\left(r,t\right)$$

respectively. Then as long as q increases from 0 to 1, the solution $\phi\left(r,t;q\right)$ varies from initial guess $u_0\left(r,t\right)$ to the exact solution $u\left(r,t\right)$.

Liao [5] by Taylor theorem expanded $\phi\left(r,t;q\right)$ in a power series of q as follows

$$\phi\left(r,t;q\right)=\phi\left(r,t;0\right)+\sum_{m=1}^{\infty}u_m\left(r,t\right)q^m, \tag{3}$$

where

$$u_m\left(r,t\right)=\frac{1}{m!}\frac{\partial^m \phi\left(r,t;q\right)}{\partial q^m}\bigg|_{q=0}. \tag{4}$$

The convergence of the series in Equation (3) depends upon the auxiliary function $\mathcal{H}\left(r,t\right)$, auxiliary parameter $\hbar$, auxiliary linear operator ℓ and initial guess $u_0\left(r,t\right)$. If these are selected properly, the series in Equation (3) is convergence at $q=1$, and one has

$$u\left(r,t\right)=u_0\left(r,t\right)+\sum_{m=1}^{\infty}u_m\left(r,t\right). \tag{5}$$

Based on Equation (2), the governing equation can be derived from the zeroth-order deformation in Equation (5) and the exact solution can be defined in vector form

$$\overrightarrow{u_n\left(r,t\right)}=\left\{u_0\left(r,t\right),u_1\left(r,t\right),\ldots,u_n\left(r,t\right)\right\}.$$

Differentiating *m*-times of zeroth-order deformation Equation (2) with respect to q and dividing them by $m!$ and also setting $q=0$, the result will be so-called *m*th-order deformation equation

$$\ell\left[u_m\left(r,t\right)-\chi_m u_{m-1}\left(r,t\right)\right]=$$
$$\hbar\mathcal{H}\left(r,t\right)\mathcal{R}_m\left(u_{m-1},r,t\right), \tag{6}$$

where

$$\chi_m=\begin{cases}0 &,m\leq 1\\1 &,m>1\end{cases}, \tag{7}$$

$$\mathcal{R}_m\left(u_{m-1},r,t\right)=\frac{1}{\left(m-1\right)!}$$
$$\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\left[\sum_{m=0}^{\infty}u_m\left(r,t\right)q^m\right]\right\}\bigg|_{q=0} \tag{8}$$

THEOREM **1** (Liao [5])**:**

The series solution (5) is convergent to the exact solution of Equation (1) as long as it is convergent.

■3.0 ANALYSIS AND TECHNIQUE OF nHAM

We transformed Equation (1) in the form as below

$$\ell u(x,t)+Au(x,t)+Bu(x,t)=0 \tag{9}$$

with initial conditions

$$u(x,0)=u_0(x,t)=f_0(x), \tag{10}$$

$$\frac{\partial u(x,t)}{\partial t}\bigg|_{t=0}=v_0(x)=f_1(x), \tag{11}$$

where $\ell=\dfrac{\partial u(x,t)}{\partial t}$ is a linear operator and $Au(x,t), Bu(x,t)$ are linear and nonlinear parts of Equation (9) respectively.

Based on HAM, the zeroth-order deformation equation is

$$(1-q)\ell[\varphi(x,t;q)-u_0(x,t)]=$$
$$q\hbar\mathcal{H}\left(r,t\right)(\ell u(x,t)+Au(x,t)+Bu(x,t)), \tag{12}$$

and the *m*th-order deformation equation is obtained as

$$\ell[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \\ \hbar \mathcal{H}(r,t)(\ell u_{m-1}(x,t) + A u_{m-1}(x,t) + B u_{m-1}(x,t)) . \tag{13}$$

By considering $\mathcal{H}(r,t) = 1$, and ℓ^{-1} as an integral operator, one has

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \\ \hbar \int_0^t \left(\frac{\partial u_{m-1}(x,t)}{\partial t} + A u_{m-1}(x,t) + B u_{m-1}(x,t) \right) dt, \tag{14}$$

for $m = 1$, $\chi_1 = 0$ and $u_0(x,0) = u_0(x,t) = f_0(x)$. The Equation (14) becomes

$$u_1(x,t) = \hbar \int_0^t (A u_0(x,t) + B u_0(x,t)) dt, \tag{15}$$

and for $m > 1$, $\chi_m = 1$ and $u_{m-1}(x,0) = 0$, it becomes

$$u_m(x,t) = (1+\hbar) u_{m-1}(x,t) + \\ \hbar \int_0^t (A u_{m-1}(x,t) + B u_{m-1}(x,t)) dt. \tag{16}$$

Now, we rewrite Equation (1) in a system of first order differential equations as

$$u_t(x,t) - v(x,t) = 0 \tag{17}$$

$$v(x,t) + A u(x,t) + B u(x,t) = 0. \tag{18}$$

From (16), (17) and (18) we have

$$u_1(x,t) = \hbar \int_0^t (-v_0(x,t)) dt \tag{19}$$

$$v_1(x,t) = \hbar (A u_0(x,t) + B u_0(x,t)), \tag{20}$$

and for $m > 1$, $\chi_m = 1$ and $u_m(x,0) = 0, v_m(x,0) = 0$, we obtain the following results

$$u_m(x,t) = (1+\hbar) u_{m-1}(x,t) + \hbar \int_0^t (-v_{m-1}(x,t)) dt \tag{21}$$

$$v_m(x,t) = (1+\hbar) v_{m-1}(x,t) + \\ \hbar (A u_{m-1}(x,t) + B u_{m-1}(x,t)) . \tag{22}$$

Equations (21) and (22) represent the general nHAM solution of Equation (9).

■**4.0 HAM SOLUTION OF KdV**

Let us consider the celebrated Korteweg-de Vries equation (KdV). This is given by

$$u_t - 6 u u_x + u_{xxx} = 0, \quad x, t \in R \tag{23}$$

subjects to the initial condition

$$u(x,0) = f(x). \tag{24}$$

We shall assume that the solution *u*(*x, t*) and its derivatives tend to zero (see [22, 23]) as |x|→ ∞.

The nonlinear KdV Equation (23) is an important mathematical model in nonlinear wave's theory and nonlinear surface wave's theory of engineering mathematics. The same examples are widely used in solid state physics, fluid physics, plasma physics, and quantum field theory ([24, 25]). The exact solution of KdV equation is given by

$$u(x,t) = -2k^2 \frac{e^{k(x-k^2 t)}}{(1+e^{k(x-k^2 t)})^2} . \tag{25}$$

For HAM solution of KdV equation we choose

$$u_0(x,t) = \frac{-2e^x}{(1+e^x)^2} \tag{26}$$

as the initial guess and

$$\ell[u(x,t;q)] = \frac{\partial u(x,t;q)}{\partial t} \tag{27}$$

as the auxiliary linear operator satisfying

$$\ell[c] = 0 \tag{28}$$

where *c* is a constant.
We consider the auxiliary function

$$\mathcal{H}(x,t) = 1 \tag{29}$$

and the zeroth-order deformation problem is given by

$$(1-q)\ell\left[u(x,t;q) - u_0(x,t)\right] = \\ q\hbar \mathcal{N}\left[u(x,t;q)\right], \tag{30}$$

$$\mathcal{N}\left[u(x,t;q)\right] = \frac{\partial u(x,t;q)}{\partial t} - \\ 6u(x,t;q)\frac{\partial u(x,t;q)}{\partial x} + \frac{\partial^3 u(x,t;q)}{\partial x^3} . \tag{31}$$

The *m*th-order deformation problem

is

$$\ell\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar[\frac{\partial u_{m-1}(x,t)}{\partial t} - \\ 6\sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-1-i}(x,t)}{\partial x} + \frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}] , \tag{32}$$

$$u_m(x,t) = 0, \quad (m \geq 1). \tag{33}$$

We have used MATHEMATICA for solving the set of linear equations (32) with condition (33). It is found that the HAM solution in a series form is given by

$$u(x,t) = \frac{-2e^x}{(1+e^x)^2} + \frac{2e^x(-1+e^x)}{(1+e^x)^3} \hbar t + \cdots \tag{34}$$

■5.0 nHAM SOLUTION OF KdV

For nHAM solution of KdV, the linear operator ℓ, the auxiliary function $\mathcal{H}(r,t)$ and initial guess function $u_0(r,t)$ are the same as for HAM solution.

We rewrite KdV equation in the form of system of equations (17) and (18) as

$$u_t(x,t) = v(x,t)\,, \quad (35)$$

$$v(x,t) = 6u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^3 u(x,t)}{\partial x^3} \quad (36)$$

and by choosing

$$v_0(x,t) = \frac{-4e^{2x}}{(1+e^x)^3} + \frac{2e^x}{(1+e^x)^2}\,, \quad (37)$$

from (19) and (20), we have

$$u_1(x,t) = \hbar \int_0^t (-v_0(x,t))dt\,, \quad (38)$$

$$v_1(x,t) = \hbar(\frac{\partial^3 u_0(x,t)}{\partial x^3} - 6u_0(x,t)\frac{\partial u_0(x,t)}{\partial t})\,. \quad (39)$$

By solving the set of linear Equations (21) and (22), it is found that the nHAM solution in a series form is given by

$$u(x,t) = \frac{-2e^x}{(1+e^x)^2} + \left(\frac{4e^x}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2} \right)\hbar t + \cdots \quad (40)$$

■6.0 HAM SOLUTION OF BURGERS' EQUATION

The Burgers' equation is described by

$$u_t + uu_x - u_{xx} = 0, \quad x,t \in R \quad (41)$$

subjects to the initial condition

$$u(x,0) = f(x)\,, \quad (42)$$

and the exact solution of this Equation is [9]

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x - \frac{1}{2}t)\,. \quad (43)$$

For HAM solution of Burgers' equation the auxiliary linear operator ℓ, the auxiliary function $\mathcal{H}$ and the zeroth-order deformation equation are the same as KdV equation. However the initial guess is taken as

$$u_0(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x)\,, \quad (44)$$

The mth-order   deformation   problem

$$\ell\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar[\frac{\partial u_{m-1}(x,t)}{\partial t} +$$

is

$$\sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-1-i}(x,t)}{\partial x} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}]\,, \quad (45)$$

$$u_m(x,t) = 0, \quad (m \geq 1)\,. \quad (46)$$

It is found that the HAM solution in a series form is given by

$$u(x,t) = \frac{1}{2} - \frac{-1+e^{\frac{x}{2}}}{2(1+e^{\frac{x}{2}})} - \frac{e^{\frac{x}{2}}(-1+e^{\frac{x}{2}})}{4(1+e^{\frac{x}{2}})^3}\hbar t + \cdots \quad (47)$$

■7.0 nHAM SOLUTION OF BURGERS' EQUATION

For nHAM solution of Burgers' equation, the linear operator ℓ, the auxiliary function $\mathcal{H}(r,t)$ and initial guess function $u_0(r,t)$ are the same as for HAM solution. By choosing

$$v_0(x,t) = \frac{1}{16}\text{sech}(\frac{x^2}{16}) \quad (47)$$

from (19) and (20), we have

$$u_1(x,t) = -\frac{1}{16}\hbar\,t\,\text{sech}(\frac{x^2}{16}) \quad (48)$$

$$v_1(x,t) = \hbar(u_0(x,t)\frac{\partial u_0(x,t)}{\partial t} - \frac{\partial^2 u_0(x,t)}{\partial x^2}) \quad (49)$$

By solving the set of linear equations (21) and (22), it is found that the nHAM solution in a series form is given by

$$u(x,t) = \frac{1}{2} - \frac{1}{16}\hbar t\,\text{sech}(\frac{x^2}{16}) -$$
$$\frac{1}{16}\hbar(1+\hbar)t\,\text{sech}(\frac{x^2}{16}) + \cdots \quad (50)$$

■8.0 RESULTS AND DISCUSSION

The approximate analytical solutions of KdV and Burgers' equations by HAM and nHAM are respectively given by (34), (40), (47) and (50), containing the auxiliary parameter $\hbar$, which influences the convergence region and rate of approximation for the HAM and nHAM solutions. In Figures 1 and 2 the $\hbar$ - curves are plotted for $u(x,t)$ of KdV and Burgers' for HAM and nHAM solutions.
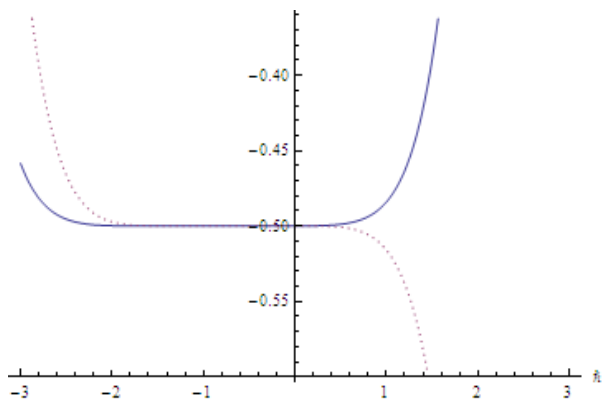
Figure 1   The $\hbar$-curves of 8th-order approximation dashed point: $u(0.01, 0.01)$ of HAM, solid line: $u(0.01, 0.01)$ of nHAM
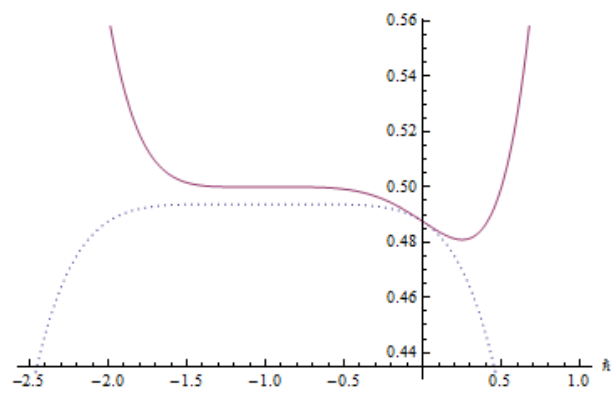

Figure 2   The $\hbar$-curves of 6th-order approximation dashed point: $u(0.1, 0.1)$ of HAM, solid line: $u(0.1, 0.1)$ of nHAM

As pointed out by Liao [4], the valid region of $\hbar$ is a horizontal line segment. It is clear that the valid region for KdV case is $-1.75 < \hbar < 0$ and for Burgers' case is $-1.4 < \hbar < -0.4$. According to theorem **1,** the series solutions (34), (40), (47) and (50) are convergent to the exact solution, as long as they are convergent. In KdV case for $-1 < t < 1$ and $\hbar = -0.4$ and in Burgers' case for $0 < t < 1$ and $\hbar = -0.5$, the results are shown to be in excellent agreement between the exact soliton solutions and the HAM and nHAM solutions. However, the results of HAM are shown to be more accurate than the nHAM solutions, as shown in Figures 3 and 4. The obtained numerical results are summarized in Tables 1 and 2. The graphs in Figures 3 and 4 upon comparing with the exact solutions, look almost the same for both cases since the errors generated by HAM and nHAM are very small.

Table 1 Comparison of the HAM and nHAM solutions with exact solution of KdV equation, when $\hbar = -0.4$

| t	x	Absolute error of exact and HAM	Absolute error of exact and nHAM
0.01	-10	$2.30751098709 \times 10^{-6}$	$7.87286655809 \times 10^{-7}$
	-6	$7.99415717104 \times 10^{-7}$	$4.30170551576 \times 10^{-5}$
	2	$2.7272317963 \times 10^{-5}$	$1.402638970418 \times 10^{-3}$
	10	$1.57356816757 \times 10^{-6}$	$7.96413206143 \times 10^{-7}$
0.05	-10	$1.72792830854 \times 10^{-4}$	$1.68759719963 \times 10^{-4}$
	-6	$3.50666740511 \times 10^{-6}$	$2.145785854685 \times 10^{-4}$
	2	$1.44338022837 \times 10^{-4}$	$7.02263452184 \times 10^{-3}$
	10	$8.8920249709 \times 10^{-8}$	$3.98885074057 \times 10^{-6}$
0.10	10	$1.0875543637 \times 10^{-7}$	7.90769×10^{-6}
	-6	$5.8990222719 \times 10^{-6}$	$4.27631300765 \times 10^{-4}$
	2	$3.07311890611 \times 10^{-4}$	$1.407815107149 \times 10^{-2}$
	10	$2.05758438852 \times 10^{-7}$	$7.98910736799 \times 10^{-6}$
0.25	-10	$1.45809251660 \times 10^{-7}$	$1.94741721345 \times 10^{-6}$
	-6	$7.95303555325 \times 10^{-6}$	$1.053732820144 \times 10^{-3}$
	2	$8.77516511341 \times 10^{-4}$	$3.56071543967 \times 10^{-2}$
	10	$7.6686337212 \times 10^{-7}$	$1.995839616844 \times 10^{-5}$
0.5	-10	$2.91478844961 \times 10^{-6}$	$3.75190695286 \times 10^{-5}$
	-6	$1.65969821206 \times 10^{-6}$	$2.03374060265 \times 10^{-3}$
	2	$1.79253970511 \times 10^{-3}$	$7.38365268199 \times 10^{-2}$
	10	$2.96635221969 \times 10^{-4}$	$3.91016008072 \times 10^{-5}$
0.75	-10	$1.41071677047 \times 10^{-7}$	$3.23130081296 \times 10^{-5}$
	-6	$2.29339604452 \times 10^{-3}$	$5.26650113038 \times 10^{-4}$
	2	$1.7658300572 \times 10^{-3}$	$1.174845846589 \times 10^{-1}$
	10	$6.75319256346 \times 10^{-6}$	$5.5674393326 \times 10^{-5}$
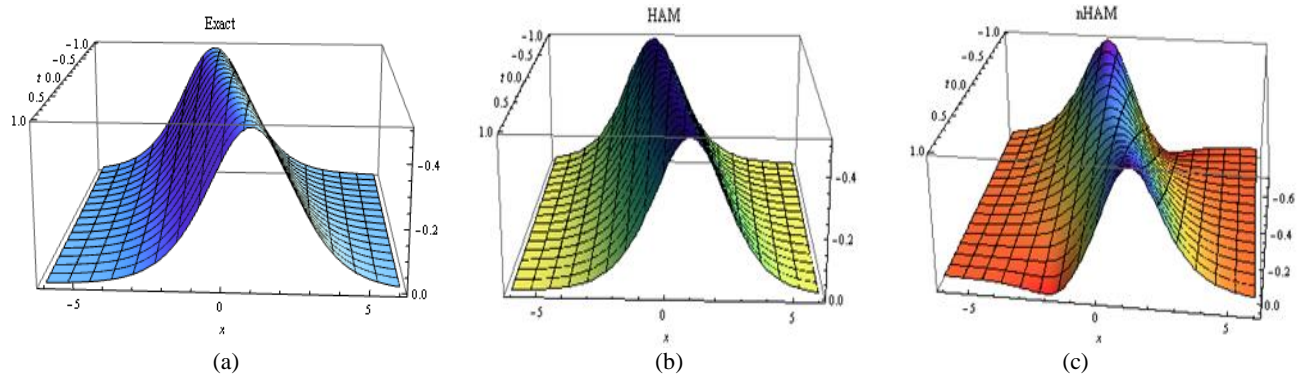
Figure 3 Comparison of the exact solution with HAM and nHAM solutions of KdV equation, when $\hbar = -0.4$
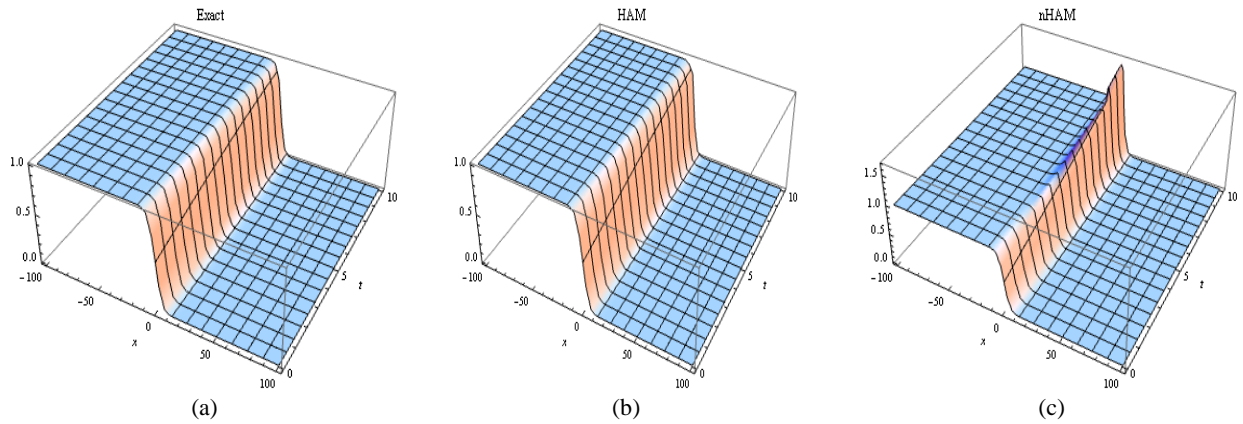

Figure 4 Comparison of the exact solution with HAM and nHAM solutions of Burgers' equation, when $\hbar = -0.5$

Table 2 Comparison of the HAM and nHAM solutions with exact solution of Burgers' equation, when $\hbar = -0.5$

t	x	Absolute error of exact and HAM	Absolute error of exact and nHAM
0	-100	0.	0.
	-50	$2.22044604925031 \times 10^{-16}$	$1.38878020194965 \times 10^{-10}$
	-20	$1.67001412698652 \times 10^{-10}$	$1.67001967810165 \times 10^{-10}$
	0	0.	0.
	20	$1.67001579232106 \times 10^{-10}$	$1.67001523632864 \times 10^{-10}$
	50	$1.11022302462515 \times 10^{-16}$	$1.11022293001637 \times 10^{-16}$
	100	0.	0.
0.25	-100	0.	0.
	-50	$4.140021658827212 \times 10^{-13}$	$1.17261755860909 \times 10^{-12}$
	-20	$1.3534540239750705 \times 10^{-6}$	$3.83454976997453 \times 10^{-6}$
	0	$7.8071276247235456 \times 10^{-3}$	$2.148436896015305 \times 10^{-2}$
	20	$1.4866585556773215 \times 10^{-6}$	$3.967864057093716 \times 10^{-6}$
	50	$4.5499239540218128 \times 10^{-13}$	$1.214084388575998 \times 10^{-12}$
	100	$1.2439367783249045 \times 10^{-23}$	0.
0.5	-100	0.	0.
	-50	$7.9025674892818 \times 10^{-13}$	$2.29649632643713 \times 10^{-12}$
	-20	$2.58351809157836 \times 10^{-6}$	$7.53630695471007 \times 10^{-6}$
	0	$1.55874519080761 \times 10^{-3}$	$4.29686926009211 \times 10^{-2}$
	20	$3.11675221511028 \times 10^{-6}$	$8.0685200420349 \times 10^{-6}$
	50	$9.5354132787939 \times 10^{-13}$	$2.46835885064599 \times 10^{-12}$
	100	$2.5681005860945 \times 10^{-23}$	$2.56810058609473 \times 10^{-23}$

t	x	Absolute error of exact and HAM	Absolute error of exact and nHAM
0.75	-100	0.	0.
	-50	$1.1315393066979 \times 10^{-12}$	$9.24771370591770 \times 10^{-12}$
	-20	$3.6987971044899 \times 10^{-12}$	$4.83746376289673 \times 10^{-12}$
	0	$2.3315894438938 \times 10^{-2}$	$1.58129338651002 \times 10^{-5}$
	20	$4.9003693713551 \times 10^{-6}$	$9.37256576276572 \times 10^{-2}$
	50	$1.4992614517953 \times 10^{-12}$	$6.81104116420638 \times 10^{-5}$
	100	$5.5511111454601 \times 10^{-17}$	$5.58347812429430 \times 10^{-12}$
1	-100	0.	$1.08159037282007 \times 10^{-11}$
	-50	$1.44007028524129 \times 10^{-12}$	$4.44844161506807 \times 10^{-12}$
	-20	$4.707292264982676 \times 10^{-6}$	$1.4541232586701 \times 10^{-5}$
	0	$3.096521597964419 \times 10^{-2}$	$8.593663034208 \times 10^{-2}$
	20	$6.849238788268118 \times 10^{-6}$	$1.666840002240 \times 10^{-5}$
	50	$2.095452595395802 \times 10^{-12}$	$5.0992543521043 \times 10^{-12}$
	100	0.	0.

■9.0 CONCLUSION

In this work, a comparative analysis of HAM and nHAM methods is implemented for the KdV and the Burgers' equations. The results obtained by HAM and nHAM are compared with the standard exact soliton solution of KdV and Burgers' equations and found to be in excellent agreement. However, the HAM solution of these equations is observed to be more accurate than the nHAM solution. We are of the opinion that this observation ratifies Hassan and El-Tawil[20] and [21], which states that the new technique of HAM, i.e. nHAM, is more suitable to obtain approximate analytical solutions of some initial value problems of high-order t-derivative (order $n \geq 2$) of nonlinear partial differential equations. However the KdV and Burgers' equations are examples where the order $n = 1$, and we have shown that the accuracy of HAM for these cases are better than nHAM.

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