GPS SATELLITE ORBIT INTEGRATION BY THE GAUSS-JACKSON PROCESS

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Abstract

High precision orbit determination Is a key requirement In preslce relative and absolute positioning by the Global Positioning System (OPS) The Gauss-Jackson 8^ Order Process using the predictor corrector scheme is used in the study to perform the orbit integration. The mathematical expressions, accuracy and efficiency of the method are discussed Results show that the Gauss-Jackson numerical procedure proved to be a precise and efficient method for the GPS orbit determination.

1.0 Introduction

Numerical integration was Introduced into dynamical astronomy by Cowell after the discovery of the eighth satellite of Jupiter The method has been applied extensively by others in investigation of the motion of satellites, such as the Halley's comet. Formulae for numerical integration have been developed since the seventeenth and eighteenth centuries by many mathematicians, such as Newton and Gregory.

Roy (1978) demonstrated that the numerical method of solving the equations of motion Is simple and fast compared to the analytical method. This view Is shared by many others, for example, Merson (1973) and Herrick (1972). The analytical solutions of the satellite equations of motion are based on complex theories of classical mechanics, truncated Infinite series and deal with large amounts of algebra Thus the direct analytical solution of the satellite perturbed motion Is not possible. For GPS satellites the best analytical theories produce orbits with an accuracy of 20 to 30 metres. King et al (1985)

In the numerical approach of solving the equations of motion, the perturbed accelerations can be modelled as accurately as they are known. Hence, there is no appoxlnation made in the solution. The availibility of high speed computer and numerical Integration software enables large number of equations to be solved with a high order of accuracy. The round-off or truncation errors that effect the accuracy of the orbits are relatively small when large and fast computers are used and the computation carried out with double precision. Hence the numerical solution will normally provide an orbit of very high precision

The Gauss-Jackson numerical formulae had been used successfully for the computation of near-earth orbiting satellites, such as the GEOS-3 and LAGEOS satellites. Theoretially the Gauss-Jackson formulae are suitable for computing GPS satellite since Its orbit is more circular than those of the GEOS-3 and LAGEOS.

In this article the numerical integration procedure of satellite orbit is reviewed. This leads to the discussion on the mathematical expressions of the Gauss-Jackson 8^{th} order process in integrating the GPS orbit. Results of the study of the error estimates of the predictor and corrector formulae, and integration step length of the method are also presented.

2.0 Numerical Integration of the Equations of Motion

In simple term orbit integration is the process of computing the satellite state vectors, ie, either the six orbital elements or in a rectangular coordinate system the satellite position and velocity vectors at the required epoch.

Numerical integration methods of evaluating the satellite equation of motion are classified into two procedures, the single-step and the multi-step method. The single-step method requires interpolation or extrapolation schemes to obtain the variables, but it does not require a "starting procedure (process of determining the solution of the differential equations at the necessary number of abscissca points to start the numericall formulae used in the integration process). The multi-step method is an iterative solution of solving the equations of motion, usually carried out using predictor corrector numerical integrator scheme. All numerical formulae require a special starting procedure before the predictor-corrector formulae can operate

Accounting for the effect of the perturbing forces, the equations of the perturbed satellite motion is given as

$$\ddot{\mathrm{r}} = \frac{Gm_e}{|r_s|^3} r_s + \ddot{\mathrm{r}}_f \qquad (1)$$

where,

r_s is the vector from the centre of mass of the earth to the centre of mass of the satellite,

$|r_s|$ is the vector length of the satellite.

m_e is the mass of the earth,

g is the universal gravitational constant,

$\ddot{r}_f$ is the total perturbing acceleration acting on the satellite

The first term in equation (1) represents the radially symmetric part of the earth's gravity field, which is the dominant term and represents the central-acceleration. The total perturbing acceleration of the separate forces which include the non-central component of the gravitational attraction of the earth, the third body effects (attraction of the sun, moon and planets), solar radiation pressure, tidal effects (solid earth tides and ocean tides), atmospheric air drag and propulsion force.

The formulation and the numerical integration of the equations of motion of a satellite are carried out in the inertial (non-rotating) reference system which is based on the fundamental astronomical system and currently maintained by IERS (International Earth Rotating Service).

In order to use numerical solution, equation (1) can be written in the form of

$$\mathrm{x}(t) = \mathrm{x}_o + \int_{t_o}^{t} \mathrm{x}\,dt \tag{2}$$

$$\dot{\mathrm{x}}(t) = \dot{\mathrm{x}}_o + \int_{t_o}^{t} \dot{\mathrm{x}}\,dt \tag{3}$$

where,

x_0 is the initial position vectors.

$\dot{x}_0$ is the initial velocity vectors and

t_0 is the time of starting epoch

The initial position and velocity vectors, known as the initial conditions, for the integration process to start must be approximately known. For GPS satellite orbit can be computed using the broadcast ephemeris. The initial conditions need to be improved and the estimation of the corrections are obtained through an iterative least squares solution using GPS observations known as orbit improvement. In the orbit improvement process the orbital parameters, corrections for the initial conditions, coordinates of the tracking stations and any perturbing parameters are estimated simultaneously.

The satellite state vectors are evaluated as a function of time by numerically integrating equations (2) and (3), either by the single-step or multi-step method. The subsequent satellité position and velocity vectors, x_1, $\dot{x}_1$, x_2, $\dot{x}_2$, etc, along the orbit are computed at discrete time according to the integration step length use.

2.1 Single -Step Method

To simplify the discussion, an independent variable is introduced via the first order differential equation.

$$\dot{s} = \frac{ds}{dt} = f(t, s) \tag{4}$$

where f(t,s) denotes some function of time, t, and s is the variable. Successive values of s can be obtained in the same way as equations (2) and (3) by integrating the function ;,

$$s(t) = s_o + \int_{t_o}^{t} f(t, s)\,dt \tag{5}$$

where s_o is the initial value of s at time t_o. Equation (5) is then evaluated numerically. In the single-step method the next value of the variable is obtained in one step by making use of the current value. The (k+1) th value of the variable is given by

$$(s_{k+1}) = s_k + \int_{t_k}^{t_k+h} f(t,s)dt \tag{6}$$

where

$$s_k = s(t_k) = s(t_o + kh) \tag{7}$$

and h is the integration step length. In solving the equations of motion by the single-step method the integral in equation (6) is evaluated using only the value of the current step, s_k. The subsequent values of the variable s_1, s_2, etc. can be computed by successive applications of the method.

2.2 Multi-step Method

As mentioned earlier, the multi-step method is an iterative process of evaluating the differential equations using predictor-corrector formulae. When an n^{th} order multi-step method is used, the predicted $(k+1)^{th}$ value of the variable, say $(s_{k+1})_1$, is obtained by a predictor formula making use the previous (n+1) value. This value is now used, along with the previous n values, to evaluate the corrected value of the variable, say $(s_{k+2})_2$, using a corrector formula. If the difference between the corrected and previous value exceeds a certain limit, the corrector formula is reapplied using the most recent available value of (s_{k+1}). The integration process is continued until the end of the interval is reached. In order to apply the multi-step method, the first (n+1) of s, ie the starting values for the integration procedure, must be determined, before any numerical formulae can be used. These values can be computed by the single-step method. Various methods of starting procedures are available, such as the Runge-Kutta method, Roy (1987), Butcher process and the Herrick method, Merson (1973).

3.0 The Gauss-Jackson Process

The Gauss-Jackson Process, alternatively known as Gauss-Jackson "Second-Sum" formulae or procedure of numerical integration was developed by Gauss and later perfected by Jackson. In this study the orbit integration is perfomed by the 8^{th} order formulae with the equations of motion formulated into the Cowell equations. The Gauss-Jackson 8^{th} order process is a multi-step approach and therefore the equations of motion must be integrated by some means for 8 steps to give a table of acceleration values at 9 points. In the study the values are determined using an iterative starting scheme, Sinclair (1987).

In the Cowell equations, the satellite second order equations of motion arise when the accelerations of the position coordinates are treated directly. The acceleration components of the satellite are then computed as the function of time using

$$\ddot{x}_i = f(t_i,\ \dot{x}_i,\ \dot{y}_i,\ \dot{z}_i,\ x_i,\ y_i,\ z_i) \tag{8}$$

$$\ddot{y}_i = f(t_i,\ \dot{x}_i,\ \dot{y}_i,\ \dot{z}_i,\ x_i,\ y_i,\ z_i) \tag{9}$$

$$\ddot{z}_i = f(t_i,\ \dot{x}_i,\ \dot{y}_i,\ \dot{z}_i,\ x_i,\ y_i,\ z_i) \tag{10}$$

where i is any integer. For simplicity only the first acceleration component of the satellite is considered in the integration process. As the Gauss-Jackson process is of the multi-step type, the procedure of evaluating the differential equations described in section 2.2 holds. The integration is carried out using an equal step-length, h, starting at a specified epoch so that

$$h = t_{k+1} - t_k \tag{11}$$

where t is the epoch, k is equal to 0, 1, 2, l and l is the order of the integration process. The full derivation of the Gauss-Jackson 8th order formulae are given in Sharif (1989). Notation f_k, x_k and $\dot{x}_k$ are used to denote the accelaration, position and velocity at step k of the integration. The Gauss-Jackson process is most convenient using the backward difference operator (∇), such that

$$\nabla f_k = f_k - f_{k-1} \tag{12}$$

$$\nabla^n f_k = \nabla^{n-1} f_k - \nabla^{n-1} f_{k-1} \tag{13}$$

The first sum and second sum operator are denoted, respectively, by ∇^{-1} and ∇^{-2} which are such that

$$f_k = \nabla^{-1} f_k - \nabla^{-1} f_{k-1} \tag{14}$$

$$\nabla^{-1} f_k = \nabla^{-2} f_k - \nabla^{-2} f_{k-1} \tag{15}$$

The acceleration f_1, f_2,.........., f_9, and the first and second sum at step 9 required to start the Gauss-Jackson formulae to integrate from step 9 to step 10 are computed by the iterative starting scheme. It is more convenient to use abbreviated notation ∇^{s}_{k} for $\nabla^{s} f_k$, especially for programing purposes. The first and second sums, respectively, are computed from the following formulae.

$$\nabla^{-1}_{i,k-1} = h^{-1}\dot{x}_{i,k} - A_0 f_{i,k} - A_1 \nabla_{i,k+1} - A_2 \nabla^2_{i,k+1}$$

$$- A_3 \nabla^3_{i,k+2} - A_4 \nabla^2_{i,k+2} - \dots \tag{16}$$

and

$$\nabla^{-2}_{i,k-1} = h^{-2} x_{i,k} - B_0 f_{i,k} - B_2 \nabla^2_{i,k+1}$$

$$- B_4 \nabla^4_{i,k+2} - B_6 \nabla^6_{i,k+3} - \dots \tag{17}$$

where the series of A and B coefficients are given in table 2. The pattern of the difference can be seen in table 1. From the known values at 9 points computed from iterative starting scheme, the right hand part of the table above the diagonal line is formed. Then the first sum at step 4, $\nabla^{-1}_{i,4}$, and second sum at step 3, $\nabla^{-2}_{i,4}$, respectively are calculated using equations (16) and (17) and the remainder of the sum above the line are completed by addition. In order to continue the table, the predictor formulae for integrating the velocity and position are required. These formulae replace the function f(t,s) in equation (5). The Gauss-Jackson predictor formulae for velocity and position, respectively, are given by

$$\dot{x}_{i,k+1} = h\left\{\nabla^{-1}_{i,k} + F_0 f_{i,k} + \sum_{j=1}^{l} F_j \nabla^j_{i,k}\right\} \tag{18}$$

and

$$x_{i,k+1} = h^2\left\{\nabla^{-2}_{i,k} + C_0 f_{i,k} + \sum_{j=1}^{l} C_j \nabla^j_{i,k}\right\} \tag{19}$$

where F and C are the coefficients, listed in table 3. The predictor formulae use the difference above the diagonal line of table 1. $\dot{x}_{i,k+1}$ and $x_{i,k+1}$ are then substituted in the accelaration farmulae to give the predicted value of $f_{i,k+1}$, and the new row of backward differences and sum is completed. If a predictor only scheme in required, the cycle is complete and the process is discribed as a PE-method. However, it is relatively fast to use corrector formulae. The Gauss-Jackson corrector formulae for velocity and position, respectively, are given by

$$\dot{x}_{i,k+1} = h\left\{\nabla^{-1}_{i,k} + E_0 f_{i,k+1} + \sum_{j=1}^{l} E_j \nabla^j_{i,k+1}\right\} \tag{20}$$

and

$$x_{i,k+1} = h^2\left\{\nabla^{-2}_{i,k} + D_0 f_{i,k+1} + \sum_{j=1}^{l} D_j \nabla^j_{i,k+1}\right\} \tag{21}$$

where E and D are the coefficients listed in table 4. After applying equations (20) and (21) the backward differences are recalculated. The procedure is considered complete at this stage and is described as a PE(CE) method. Since the applications of the corrector formulae is relatively fast, it is convenient to apply equation (20) and (21) the second time. The latest corrected value of $f_{i,k+1}$ compared with the previous value. If the difference exceeds a certain limit, the corrector formulae are reapplied using the most recent values of the variables until the accuracy required is satisfied.

3.1 Error Estimates Of The Method

The accuracy of the convengence of the iteration is estimated by recording the maximum difference between any of the three components of the position vectors for the penultimate and final iteration. Therefore this gives the actual convengence error of the penultimate iteration, and is a pessimistic estimate of the final iteration.

The error estimates of the formulae are obtained from the integration using the final corrected initial conditions. In this example the orbit is computed for one satellite revolution, ie. 12 hours. The improvement stage of computing the starting values is repeated as many times as specified. At least 3 iterations are needed for reasonable accuracy, and 5 iterations will usually give more than sufficient accuracy.

The following error estimates are abtained when the integration is carried out using integration step length of 5 minutes interval.

i. The maximum truncation error of formulae for position is $0.80\text{x}10^{-9}$ metres.

ii. The maximum convergence error of penultimate iteration is $0.71\text{x}10^{-8}$ metres.

iii. The estimated convergence error of the final iteration is $0.43\text{x}10^{-9}$ metres.

iv. The maximum truncation error of formulae for velocities is $0.43\text{x}10^{-8}$ metres per second.

v. The convergence error of penultimate iteration for the partial derivatives of position with respect to initial position is $0.17\text{x}10^{-7}$ (dimensionless).

vi. The convergence error of penultimate iteration for the partial derivatives of position with respect to initial velocity in $0.15\text{x}10^{-20}$ seconds

The accuracy of the integrated orbit in this case is determined by comparing the integrated coordinates with the precise ephemenides obtained from the Centre for Space Research, University of Texas, USA (CSR/UT). These ephemerides are expected to be accurate to 1 to 2 metres.

3.2 Analysis of Integration Step Length

In numerical orbit integration the size of the integration must be able to cope with the perturbations caused by the high order tesseral components of the gravity field. However the high altitude of the GPS satellites enables the use of high order integration formulae and large integration step length. In this investigation various sizes of integration step length ranging from 2 to 30 minutes were tested. The GPS orbits were computed for 7 days.

The results show that the error estimate increases with the size of the integration step length, showing a significant value when the integration step length is 20 minutes and above. Step length of 15 minutes and below were able to produce orbit accuracy of about 0.25 metres, figure 1. The error plots are very similar to those of the satellite clock frequency offsets, where the peak to peak frequency variations are due to the thermal cycling of the satellite clock.

Experiments also show that the accuracy of the integrated orbit decreases with orbit length. For example, using an integration step length of 20 minutes, the orbit accuracy decreases from 0.5 metres for 1 day orbit to 3.5 metres for 7 day orbit figure 2. The errors in the along-track and radial components clearly indicate the presence of both short and long period perturbations, while the cross-track components indicates only short period perturbation. The short period perturbation, 12 hours, are due to tidal effects of the sun and moon which accur twice a day. The periods of the perturbations remain constant for the 7 day orbit while the amplitude increases with the orbit length. The study shows for long orbit computation an ontegration step length of more than 20 minutes interval could not cope with the short and long perturbations caused by the high order tesseral components of the gravity field.

4.0 Conclusions

Based on the investigations of the Gauss-Jackson 8^{th} order numerical formulae for orbit integration, the following conclusions on the method are made.

a) Since the Gauss-Jackson process is a multi-step method of numerical integration, it requires initial conditions, as well as starting values. The GPS broadcast ephemeris provides a good approximation for computing the values for the initial conditions and, consequently, only few iterations are required in the orbit improvement process.

b) The orbital parameters can be determined from satellite observations using the variational equations. Since among other factors, the initial conditions determine the accuracy of the computed orbit, if they are poorly estimated the orbit improvement process using the observations has to be iterated several times.

c) With the advent of modern computers, the numerical solution is an efficient and fast process of solving the equations of motion. The use of the corrector formulae is relatively fast, since the number of interation at each integrating step to obtain a certain specified accuracy is reduced. Large integrating step length can be used and, consequently, it reduces the integrating time to a large extend for the specified orbit length. For example, the computation of a GPS orbit length up to 7 days an integration step length of 15 minutes is sufficient to produce an orbit with an accuracy of about 0.25 metres.

d) The numerical solution is a precise method of orbit determination since there is no approximation in the solution. The perturbing forces affecting the satellite motion can be modelled as accurately as they are known.

References

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Argument		Funct.	Differences							
			1st	2nd	3rd	4th	5th	6th	7th	8th
		$f_{i,0}$								
	$\nabla^{-1}_{i,0}$		$\nabla_{i,1}$							
		$f_{i,1}$		$\nabla^{2}_{i,2}$						
	$\nabla^{-1}_{i,1}$		$\nabla_{i,2}$		$\nabla^{3}_{i,3}$					
		$f_{i,2}$		$\nabla^{2}_{i,3}$		$\nabla^{4}_{i,4}$				
	$\nabla^{-1}_{i,2}$		$\nabla_{i,3}$		$\nabla^{3}_{i,4}$		$\nabla^{5}_{i,5}$			
		$f_{i,3}$		$\nabla^{2}_{i,4}$		$\nabla^{4}_{i,5}$		$\nabla^{6}_{i,6}$		
	$\nabla^{-1}_{i,3}$		$\nabla_{i,4}$		$\nabla^{3}_{i,5}$		$\nabla^{5}_{i,6}$		$\nabla^{7}_{i,7}$	
$\nabla^{-2}_{i,3}$		$f_{i,4}$		$\nabla^{2}_{i,5}$		$\nabla^{4}_{i,6}$		$\nabla^{6}_{i,7}$		$\nabla^{8}_{i,8}$
	$\nabla^{-1}_{i,4}$		$\nabla_{i,5}$		$\nabla^{3}_{i,6}$		$\nabla^{5}_{i,7}$		$\nabla^{7}_{i,8}$	
$\nabla^{-2}_{i,4}$		$f_{i,5}$		$\nabla^{2}_{i,6}$		$\nabla^{4}_{i,7}$		$\nabla^{6}_{i,8}$		$\nabla^{8}_{i,9}$
	$\nabla^{-1}_{i,5}$		$\nabla_{i,6}$		$\nabla^{3}_{i,7}$		$\nabla^{5}_{i,8}$		$\nabla^{7}_{i,9}$	
$\nabla^{-2}_{i,5}$		$f_{i,6}$		$\nabla^{2}_{i,7}$		$\nabla^{4}_{i,8}$		$\nabla^{6}_{i,9}$		
	$\nabla^{-1}_{i,6}$		$\nabla_{i,7}$		$\nabla^{3}_{i,8}$		$\nabla^{5}_{i,9}$			
$\nabla^{-2}_{i,6}$		$f_{i,7}$		$\nabla^{2}_{i,8}$		$\nabla^{4}_{i,9}$				
	$\nabla^{-1}_{i,7}$		$\nabla_{i,8}$		$\nabla^{3}_{i,9}$					
$\nabla^{-2}_{i,7}$		$f_{i,8}$		$\nabla^{2}_{i,9}$						
	$\nabla^{-1}_{i,8}$		$\nabla_{i,9}$							
$\nabla^{-2}_{i,8}$		$f_{i,9}$								
	$\nabla^{-1}_{i,9}$									
$\nabla^{-2}_{i,9}$										

Table 1 : Sum and Difference Table for the Gauss Jackson Eight Order Process

j	A_j
0	- 1/2
1	- 1/2
2	1/24
3	11/720
4	- 11/1440
5	- 191/60480
6	191/120960
7	2497/3628800
8	- 2497/7257600
9	- 14797/95800320
10	14797/191600640
11	92427157/2615348736000
12	- 92427157/5230697472000

j	B_j
0	1/2
1	0
2	- 1/240
3	0
4	31/60480
5	0
6	289/3628800
7	0
8	317/22809600
9	0
10	- 6803477/2615348736000
11	0
12	3203699/6276836966400

Table 2: Coefficients for Gauss-Jackson Formulae

j	C_j
0	1/12
1	1/12
2	19/240
3	3/40
4	863/12096
5	275/4032
6	33953/518400
7	8183/129600
8	3250433/53222400
9	4671/78848
10	13695779093/237758976000
11	2224234463/39626496000
12	132282840127/2414168064000

j	F_j
0	1/2
1	5/12
2	3/8
3	251/720
4	95/288
5	19087/60480
6	5257/17280
7	1070017/3628800
8	25713/89600
9	26842253/95800320
10	4777223/17418240
11	703604254357/2615348736000
12	106364763817/402361344000

Table 3: Coefficients for Gauss-Jacksonn Formulae

j	D_j	j	E_j
0	1/2	0	- 1/2
1	0	1	- 1/12
2	- 1/240	2	- 1/24
3	- 1/240	3	- 19/720
4	- 221/60480	4	- 3/160
5	- 19/6048	5	- 863/60480
6	- 9829/3628800	6	- 275/24192
7	- 407/172800	7	- 33953/3628800
8	- 330157/159667200	8	- 8183/1036800
9	- 24377/13305600	9	- 3250433/479001600
10	- 4281164477/2615348736000	10	- 4671/788480
11	- 70074463/47551795200	11	- 13695779093/2615348736000
12	- 1197622087/896690995200	12	- 2224234463/47551795200

Table 4: Coefficients for Gauss-Jackson Formulae

Figure 1

Orbit errors using integration step length of 15 min. interval

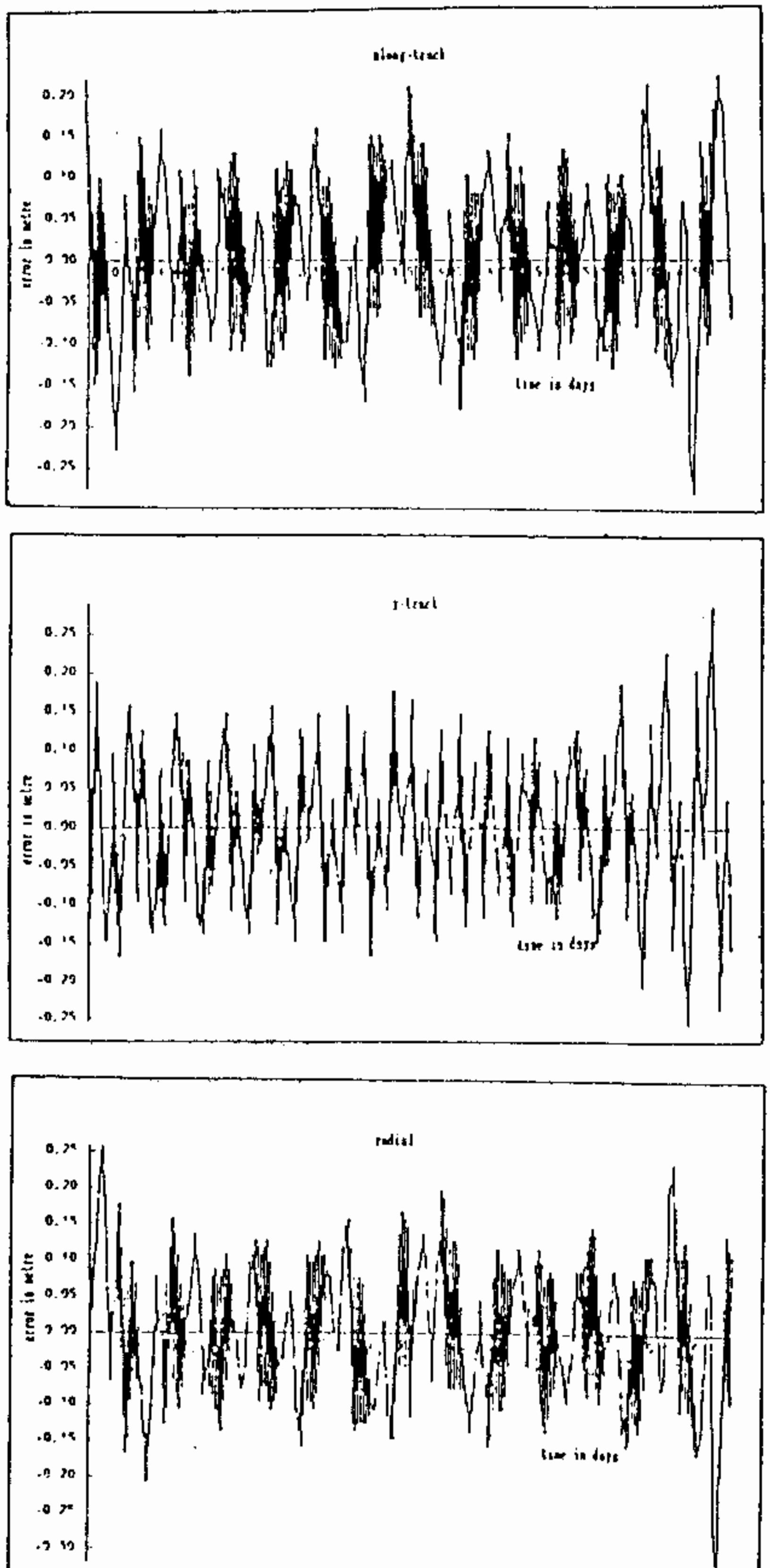

Figure 2

Orbit errors using integration step length of 20 min. interval