MIXED FORMULATION FOR NAVIER STOKES EQUATIONS

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To my mother, father, wife, daughters and son...

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ABSTRACT

This dissertation presents the formulation and numerical computation of the two dimensional fluid flow whose physics is governed by the Navier Stokes equations. The derivation of the governing equations follows the model of an infinitesimally small element fixed in space with the fluid moving through it. The study focuses using finite element method for solving the fluid flow. Finite element method is believed as the most powerful numerical method for analysis. Mixed formulation has been used for the discretization of the governing equations. It is because we have two primitive variables; velocities and pressure. Source code for Navier Stokes equations has been developed and tested against existing benchmark solutions.

ABSTRAK

Disertasi ini bertujuan menunjukkan formulasi dan kiraan berangka untuk aliran dua dimensi bendalir di mana keadaan fizik aliran tersebut ditentukan atau ditadbir oleh persamaan *Navier Stokes*. Penerbitan persamaan pentadbir ditunjukkan dengan menggunakan model elemen yang amat kecil tetap di dalam ruang dengan cecair yang bergerak melaluinya. Kajian ini memfokuskan kepada kaedah unsur terhingga bagi menyelesaikan bendalir dinamik (aliran cecair). Kaedah unsur terhingga ini dipercayai sebagai kaedah berangka yang terbaik untuk analisis. Formulasi campuran telah digunakan untuk diskretasi persamaan pentadbir kerana terdapat dua pembolehubah primitif iaitu halaju dan tekanan. Kod sumber pengaturcaraan untuk persamaan *Navier Stokes* telah dibangunkan dan diuji dengan penyelesaian penanda aras sedia ada.

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LIST OF SYMBOLS

SYMBOLS :

m	-	Mass
p	-	Pressure
t	-	Time
$\vec{V}$	-	Velocity vector
u	-	Velocity in x component
v	-	Velocity in y component
w	-	Velocity in z component
ρ	-	Density
$\frac{\partial}{\partial t}$	-	Time rate of change
$\frac{D}{Dt}$	-	Material derivative (Substantial derivative)
τ	-	Viscous stress tensor
μ	-	Molecular viscosity coefficient
λ	-	Second viscosity coefficient
$V.\nabla$	-	Convective derivative
F	-	Force on the fluid
a	-	Acceleration
B( )	-	Linear differential operator
$\{u\}$	-	Unknown function
Ω	-	Geometrical domain
α_i	-	General parameters
$P_i(\{x\})$	-	Polynomial base.

Re	-	Reynolds Number
V_∞	-	Reference speed
L	-	Reference length
f_i	-	Force on *i* direction
$w_i(x.y)$	-	Weighting function
$N_i(x.y)$	-	Quadratic shape function
$L_i(x.y)$	-	Linear shape function
(ξ,η)	-	Natural coordinates

LIST OF ABBREVIATIONS

ABBREVIATION :

CFD	-	Computational fluid dynamics
PDE	-	Partial differential equations
ODE	-	Ordinary differential equations
FVM	-	Finite volume method
FEM	-	Finite element method
FDM	-	Finite difference method
dof	-	Degree of freedom
WRM	-	Weighted residual method
GWRM	-	Galerkin weighted residual method

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Fluid mechanics is one of the branches in physics that study fluids (which consists of liquids, gases and plasmas) and the forces on them. Fluid mechanics can be divided into three categories. The first one is fluid statics (the study of fluids at rest), fluid kinematics (the study of fluids in motion) and the last one is fluid dynamics (the study of the effect of forces on fluid motion). Fluid dynamics is an active field of research with many unsolved or partly solved problems. Fluid mechanics are mathematically complex and can best be solved by numerical methods typically by using computers. A modern discipline, called computational fluid dynamics (CFD), is devoted to this approach to solving fluid mechanics problems.

One of the famous equations in CFD is Navier–Stokes equations; named after Claude-Louis Navier and George Gabriel Stokes describes the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion. The assumption is that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term, hence describing viscous flow.

The equations are very useful because they describe many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier–Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution and many other things.

The Navier–Stokes equations are also of great interest in a purely mathematical sense. Surprisingly, given their wide range of practical uses, mathematicians have not yet proven that, in three dimensions, solutions always exist (existence), or that if they do exist, then they do not contain any singularity (smoothness). These are called the Navier–Stokes existence and smoothness problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example.

The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science. Historically, the method has taken place since early 1950s in coincidence with the development of digital computers. However, interest in approximate solutions dates as far back in time as the development of the classical field theories (e.g. elasticity, electro-magnetism) themselves. The work of Lord Rayleigh (1870) and W. Ritz (1909) on variational methods and the weighted-residual approach taken by B. G. Galerkin (1915) and others form the theoretical framework to the finite element method. With a bit of a stretch, one may even claim that Schellbach's approximate solution to Plateau's problem (to find a surface of minimum area enclosed by a given closed curve in three dimensions), which dates back to 1851 is an elementary application of the finite element method.

Most researchers agree that the era of the finite element method begins with a lecture presented in 1941 by R. Courant to the American Association for the Advancement of Science. In his work, Courant used the Ritz method and introduced the pivotal concept of spatial discretization for the solution of the classical torsion problem. Courant did not pursue his idea further, since computers were still largely unavailable for his research.

More than a decade later, Ray Clough Jr. of the University of California at Berkeley and his colleagues essentially reinvented the finite element method as a natural extension of matrix structural analysis and published their first work in 1956. He attributes the introduction of the term “finite element” to M.J. Turner, one of his associates at that time. An apparently simultaneous effort by John Argyris at the University of London independently led to another successful introduction of the method. To a large extent, the finite element method appears to owe its reinvention to structural engineers. In fact, the consideration of a complicated system as an assemblage of simple components (elements) on which the method relies is very natural in the analysis of structural systems.

Few years after its introduction to the engineering community, the finite element method has attracted the attention of applied mathematicians, particularly those who is interested in numerical solution of partial differential equations. In 1973, G. Strang and G.J. Fix authored the first conclusive treatise on mathematical aspects of the method, focusing exclusively on its application to the solution of problems emanating from standard variational theorems.

The finite element has been subject to extreme research. By the beginning of the 1990s, the method clearly dominated the numerical solution of problems in the fields of structural analysis, structural mechanics and solid mechanics. Moreover, the finite element method currently competes in popularity with the finite difference method in the areas of heat transfer and fluid mechanics.

1.2 Problem Statement

There are a lot of methods to solve fluid equations which require us to understand its governing equations. These governing equations are in the form of partial differential equations (PDEs) which are differs from ordinary differential equations (ODEs) in the way that they have two or more independent variables. It is very tedious and impractical to solve PDEs analytically, though is not impossible. To that end, various numerical approaches are available to give approximate solutions.

The application of numerical methods to solve problems involving fluid flow is known as computational fluid dynamics (CFD). Most of the previous studies on CFD are using finite difference, finite volume and finite element methods. One of the major difficulties in solving fluid flow problem is the handling of pressure terms in which singularity occurs and the stability of the time integration in obtaining a converged solution. Among available techniques, mixed formulation which is the subset of finite element method has been shown to be the most appropriate. Therefore, the goal of this study is to use finite element method with mixed formulation to solve Navier Stoke equations.

1.3 Aim and Objectives

The purpose for this study is to solve the Navier Stokes equation using a finite element method with mixed formulation. The objectives of this study are:

i. To use mixed formulation for solving fluid flow equations (Navier Stokes equations).
ii. To develop the MATLAB program for (i).
iii. To verify and validate the derived formulation with CFD benchmark problems (Lid driven cavity flow).

1.4 Scope of Research

The scope of this study is on numerical simulations for fluid flow. Therefore, for this project, we will use Navier Stokes Equation. Finite element method with the mixed formulation is used as the numerical technique. The fluid is assumed to be Newtonian, viscous, incompressible, isothermal and electrically nonconducting.

1.5 Significance of Study

This study is only focus on the mixed finite element in the fluid dynamics. Other numerical technique has been done for the past decades, but not by using finite element method. May be, there are setbacks which hinder the efficiency of the solution process. Hopefully, with the approach of this study will increase the understanding on the behaviour of the flow.

1.6 Overview of Thesis

This chapter gives an introduction of Navier Stokes equation and some of the application widely used of the equations. In problem statement, the problem that the research will address is highlighted. This is followed by the aim and objectives of the research. The scope and significance of the research is also highlighted at the end of the chapter.

For chapter two, the literature review is discussed. It begins with a brief discussion for Navier Stokes equations which consist of conservation of mass, momentum and energy. A number of numerical method typically used to solve Navier Stokes equation are highlighted. Lastly, the mixed formulation for the solution of Navier Stokes equation is presented.

In chapter three, the derivation of Navier Stokes equation is presented and discussed. It starts with the derivation of continuity equation. There are four models that can be discussed. There are the finite control volume fixed in space, the finite control volume moving with the fluid, infinitesimal fluid element fixed in space with fluid moving through it and infinitesimal fluid element moving along a streamline. For this, we will use the model of small element fixed in space. Next, the momentum equation is derived where the infinitesimally small and moving fluid element will be used to derive the x component of momentum equation. The Navier Stokes equation in conservation and non conservation is also discussed an presented.

In chapter four, the mixed formulation for Navier Stokes is discussed. It begins with the discussion of Galerkin weighted residual method. Next, the dimensionless of the equation is discussed. Subsequently, the discretization of the equations from continuous to matrix and vector form is presented. Then the finite element method formulation of the equation is discussed. After that, the discretization of the equations by Galerkin weighted residual method is introduced and discussed briefly. This will be followed by time integration with the basic concept of iterative scheme. Lastly, Piccard nonlinear method is presented.

In chapter five, the verification of the code were done by the benchmark case where lid driven cavity flow is considered. The results are compared and discussed with the result from the commercial software.

Chapter six summarized the mixed formulation for Navier Stokes equations. The results from the programming are concluded and several recommendations for future works are suggested.

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