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Hirota-Sato Formalism on Some Nonlinear Waves Equations

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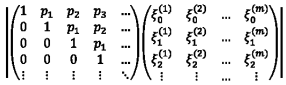

Abstract

This article demonstrates that Hirota's direct method or scheme for solving nonlinear waves equation is linked to Sato theory, and eventually resulted in the Sato equation. This theoretical framework or simply the Hirota-Sato formalism also reveals that the τ – function, which underlies the analytic form of soliton solutions of theses physically significant nonlinear waves equations, shall acts as the key function to express the solutions of Sato equation. From representation theory of groups, it is shown that the τ – function in the bilinear forms of Hirota scheme are closely connected to the Plucker relations in Sato theory. Thus Hirota-Sato formalism provides a deeper understanding of soliton theory from a unified viewpoint. The Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera equations are used to verify this framework.

Keywords: Hirota-Sato Formalism; τ – function; Plucker relations; Kadomtsev-Petviashvili (KP) equation; Korteweg-de Vries (KdV) equation; Sawada-Kotera equation

Abstrak

Artikel ini mempamerkan kaedah langsung Hirota atau skema bagi menyelesaikan persamaan tak linear gelombang itu adalah berhubungan dengan teori Sato, dan akhirnya terhasil persamaan Sato. Kerangka berteori ini atau mudahnya formalisme Hirota-Sato menyerlahkan fungsi τ, yang mendasari bentuk analitik penyelesaian soliton bagi persamaan gelombang tak linear berkepentingan fizikal itu, bertindak sebagai fungsi utama bagi mengungkap penyelesaian persamaan Sato. Daripada teori perwakilan kumpulan, dibuktikan bahawa fungsi τ dalam bentuk bilinear skema Hirota adalah berkait rapat dengan hubungan Plucker dalam teori Sato. Oleh itu, formalisme Hirota-Sato menyediakan satu kefahaman mendalam terhadap teori soliton daripada satu pendekatan menyatukan. Persamaan Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) dan Sawada-Kotera digunakan untuk mengesahkan kerangka itu.

Kata kunci: Formalisme Hirota-Sato; fungsi τ; hubungan *Plucker*; persamaan *Kadomtsev-Petviashvili (KP)*; persamaan *Korteweg-de Vries (KdV)*; persamaan *Sawada-Kotera*

1.0 INTRODUCTION

There are several methods of studying the integrable nonlinear waves equations that have soliton solutions, where each technique has its own suppositions and areas of usage. For example, the inverse scattering transform (IST) can be used to solve initial value problems, but it uses powerful analytical methods and quantum scattering theory (e.g. [1]), and therefore makes strong assumptions about the nonlinear equations. On the lesser extreme, one can find a travelling wave solution to almost all equations by a simple substitution which reduces the equation to an ordinary differential equation (e.g. [2]). Between these two extremes lies Hirota's direct/bilinear method. Although the transformation was intrinsically inspired by IST, Hirota's method does not need the same mathematical assumption and, as a consequence, the method is applicable to a wider class of equations than IST (e.g. [3]). At the same time, because it does not use such sophisticated techniques, it usually produces a smaller class of solutions, the multi-soliton solutions. It is particularly efficient for constructing multisoliton solutions to integrable nonlinear waves equations. The advantage of it over others is that it is algebraic rather than analytic. In many problems the key to further developments is a detailed understanding of soliton scattering, and in such cases Hirota's method is the optimal tool.

Hirota's method is an effective tool as it can be employed without a deep knowledge of the mathematics that lies beneath,

namely Sato's theory. This is not to say that what lies beneath is valueless, in fact in this article, we will establish that Sato's theory allows Hirota's method to have a deeper and beautiful understanding of soliton theory from a unified viewpoint. Here we note that Hirota's method makes an efficient tool for an applied mathematicians' toolbox, and furthermore, we may credit it (along with the pioneering IST) for inspiring a departure into soliton theory.

This article also develops a brief introduction to Sato's theory in an elementary way, which will serve as a departure point into soliton theory. In all of its beauty, Sato's theory connects solitons and infinite dimensional Grassmanians, which are Sato's formal generalisation of the finite dimensional Grassmanians of algebraic geometry (e.g. [4-5]). We do not attempt to reach quite as far as this inference, but aim to give an overture accessible to a researcher of applied mathematics. Even in this preamble of the framework, the beauty and power of Hirota-Sato formalism becomes apparent.

Hirota direct method was first introduced by Hirota in his well-known 1971 paper [6]. The first step of this method is to transform the nonlinear partial differential equation into a quadratic form in dependent variables. The new form of the equation is called 'bilinear form'. In the second step, we write the bilinear form of the equation as a polynomial of a special differential operator known as Hirota D-operator. This polynomial of D-operator is called 'Hirota bilinear form'. In this article we shall only focus in solving on some physically significant nonlinear wave equations which includes KP [7], KdV [8] and Sawada-Kotera equations [9]. The KdV and KP equations have been used extensively as a model for one- and two-dimensional shallow water waves of long wavelength with weakly non-linear restoring forces (e.g. [10]) and ion-acoustic waves in plasmas (e.g. [11]), and Sawada-Kotera in some mathematical approaches to tsunami (e.g. [12])

Grassmannian manifolds are known as the basics of Sato's theory where the τ-function was obtained from the derivation of the Sato's equation (e.g. [13-14]). The manipulations of Schur functions and Young diagrams later produce the Plucker relations. In this article we show how the Plucker relations in Sato's theory can be transformed into the Hirota bilinear form of KP, KdV [15], and Sawada-Kotera equations [16], [17] for their respective τ-functions, and thus verify this conceptual framework.

2.0 THEORY AND METHOD

2.1 Results Related to Hirota's Direct/Bilinear Method

To illustrate the statement mentioned in the introduction, we consider in the context of Hirota's method, some results linked to nonlinear partial differential equations related to Kadomtsev-Petviashvili (KP), Korteweg de Vries (KdV) and Sawada-Kotera equations respectively.

a) The KP equation is given by

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0. \tag{2.1}$$

The logarithmic (or dependent variable) transformation, via Hirota's method is given by

$$u = (\log\ \tau)_{xx}. \tag{2.2}$$

Thus, the Hirota bilinear form of KP takes the form of

$$\left(4D_xD_t - D_x^4 - 3D_y^2\right)\tau\,.\,\tau = 0. \tag{2.3}$$

b) The KdV equation is given by

$$u_t + 6u\,u_x + u_{xxx} = 0. \tag{2.4}$$

The logarithmic transformation, via Hirota's method is given by

$$u = 2(\log\ \tau)_{xx}. \tag{2.5}$$

The Hirota bilinear form of KdV is then given by

$$D_x(D_t + D_x^3)\,\tau\ \cdot\tau = 0. \tag{2.6}$$

c) The fifth-order Sawada-Kotera equation can be written as

$$u_t + 45u^2u_x + 15(u_xu_{xx} + u_{xxx}u) + u_{xxxxx} = 0. \tag{2.7}$$

The logarithmic transformation is the same as in (2.5), i.e.

$$u = 2(\log\ \tau)_{xx}.$$

Thus, the Hirota bilinear form of Sawada-Kotera is of the form

$$D_x(D_t + D_x^5)\,\tau\,.\,\tau = 0. \tag{2.8}$$

2.2 Sato's Theory

Let W be a pseudo-differential operator,

$$W = 1 + w_1\,\partial^{-1} + w_2\,\partial^{-2} + w_3\,\partial^{-3} + \cdots, \tag{2.9}$$

where w_j (j=1,2..., m, …) are functions of x and ∂^{-n} is defined by

$$\partial^{-n} = \left(\frac{\partial}{\partial x}\right)^{-n}. \tag{2.10}$$

The inverse operator W^{-1} exists and can be written as

$$W^{-1} = v_0 + v_1\,\partial^{-1} + v_2\,\partial^{-2} + v_3\,\partial^{-3} + \cdots, \tag{2.11}$$

where

$$v_0 = 1, \tag{2.12a}$$

$$v_1 = -w_1, \tag{2.12b}$$

$$v_2 = -w_2 + w_1^2, \tag{2.12c}$$

$$v_3 = -w_3 + 2w_1\,w_2 - w_1\,w_1^{'} - w_1^3, \tag{2.12d}$$

We introduce the term $H(x;t)$ as

$$H(x;t) = \begin{pmatrix} h_0^{(1)} & h_0^{(2)} & \dots & h_0^{(m)} \\ h_1^{(1)} & h_1^{(2)} & \dots & h_1^{(m)} \\ h_2^{(1)} & h_2^{(2)} & \dots & h_2^{(m)} \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}, \tag{2.13}$$

where equation (2.13) is the denominator of the function $W_m(x;t)$.

Let us consider the partial differential equation which depends on x and t as,

$$W_m(x;t)\partial^m h_0^{(j)}(x;t) = (\partial^m + w_1(x;t)\,\partial^{m-1} + \cdots + w_m(x;t))h_0^{(j)}(x;t) = 0 \tag{2.14}$$

where $j = 1,2,\ldots,m$, and

$$W_m(x;t) = \frac{\begin{vmatrix} h_0^{(1)} & \ldots & h_0^{(m)} & \partial^{-m} \\ \vdots & \ldots & \vdots & \vdots \\ h_{m-1}^{(1)} & \ldots & h_{m-1}^{m} & \partial^{-1} \\ h_m^{(1)} & \ldots & h_m^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} h_0^{(1)} & \ldots & h_0^{(m)} \\ \vdots & \ldots & \vdots \\ h_{m-1}^{(1)} & \ldots & h_{m-1}^{(m)} \end{vmatrix}}.$$

After differentiating (2.14) with respect to t_n and solving the remaining equation, we have

$$B_n = (W\,\partial^n W^{-1})^+, \tag{2.15}$$

where $(\,)^+$ denotes the differential part of the operator. Equation (2.15) is called the Sato equation. It is important to note that in the derivation of the Sato equation, we have assumed that the solutions take the form $W_m(x;t)$ and this will eventually yield the τ-function [4].

Let us define the τ-function as the determinant of $H(x;t)$, the denominator of the function $W_m(x;t)$,

$$\tau(x;t) = \begin{vmatrix} h_0^{(1)} & \ldots & h_0^{(m)} \\ h_1^{(1)} & \ldots & h_1^{(m)} \\ \vdots & \ldots & \vdots \\ h_{m-1}^{(1)} & \ldots & h_{m-1}^{(m)} \end{vmatrix}$$

$$= \left| \begin{pmatrix} 1 & p_1 & p_2 & p_3 & \ldots \\ 0 & 1 & p_1 & p_2 & \ldots \\ 0 & 0 & 1 & p_1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \ldots & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \ldots & \xi_1^{(m)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \ldots & \xi_2^{(m)} \\ \vdots & \vdots & \ldots & \vdots \end{pmatrix} \right|$$

$$= det\left(\Xi_0^t\, e^{\eta(t,\Lambda)}\, \Xi\right), \tag{2.16}$$

where Ξ_0^t is a $m \times \infty$ matrix defined by

$$\Xi_0^t = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.17}$$

By using the expansion theorem on the determinant of product of matrices, $\tau(t)$ in equation (2.16) can be expanded as a sum of products of determinants,

$$\tau(t) = \sum_{0\le l_1<l_2<\cdots<l_m} \begin{vmatrix} p_{l_1} & p_{l_2} & \ldots & p_{l_m} \\ p_{l_1-1} & p_{l_2-1} & \ldots & p_{l_m-1} \\ \vdots & \vdots & \ldots & \vdots \\ p_{l_1-m+1} & p_{l_2-m+1} & \ldots & p_{l_m-m+1} \end{vmatrix} \times \begin{vmatrix} \xi_{l_1}^{(1)} & \xi_{l_1}^{(2)} & \ldots & \xi_{l_1}^{(m)} \\ \xi_{l_2}^{(1)} & \xi_{l_2}^{(2)} & \ldots & \xi_{l_2}^{(m)} \\ \vdots & \vdots & \ldots & \vdots \\ \xi_{l_m}^{(1)} & \xi_{l_m}^{(2)} & \ldots & \xi_{l_m}^{(m)} \end{vmatrix}, \tag{2.18}$$

where the summation comprised of all possible combinations of m nonnegative numbers. It is also known that the determinants, composed of p_j's in equation (2.16), are the Schur functions [5]. We may denote this by

$$S_Y(t) = \begin{vmatrix} p_{l_1} & p_{l_2} & \ldots & p_{l_m} \\ p_{l_1-1} & p_{l_2-1} & \ldots & p_{l_m-1} \\ \vdots & \vdots & \ldots & \vdots \\ p_{l_1-m+1} & p_{l_2-m+1} & \ldots & p_{l_m-m+1} \end{vmatrix}, \tag{2.19}$$

and

$$\xi_Y = \begin{vmatrix} \xi_{l_1}^{(1)} & \xi_{l_1}^{(2)} & \ldots & \xi_{l_1}^{(m)} \\ \xi_{l_2}^{(1)} & \xi_{l_2}^{(2)} & \ldots & \xi_{l_2}^{(m)} \\ \vdots & \vdots & \ldots & \vdots \\ \xi_{l_m}^{(1)} & \xi_{l_m}^{(2)} & \ldots & \xi_{l_m}^{(m)} \end{vmatrix}, \tag{2.20}$$

where the suffix Y stands for the Young diagram that corresponds to the set of numbers $(l_1, l_2, \ldots, l_m)$. The Young diagram is introduced to classify the irreducible representation of the symmetric group (e.g. [18-19]). It is noted that, although different sets of numbers may correspond to a certain Y if m is not fixed; the RHS of equation (2.19) gives the same function for those sets.

Hence, equation (2.18) can be written as

$$\tau(t) = \sum_{\emptyset\le Y\le m} S_Y(t)\xi_Y, \tag{2.21}$$

where the summation includes all the Young diagrams which have less than $m+1$ rows. For the coefficients, ξ_Y's in equation (2.20), there exist constraints that are called the Plucker relations which ξ_Y's must always satisfy. This is defined in the next section.

3.0 RESULTS AND DISCUSSIONS

First, let us define the Plucker relations in the form of

$$\sum_{i=1}^{m+1}(-1)^{\delta}\,\xi_{Y_1}\,\xi_{Y_2} = 0, \tag{3.1}$$

$$\delta = m + i - j, \tag{3.2}$$

where Y_1 is the Young diagram corresponding to $(k_1,\ldots,k_j,l_i,k_{j+1},\ldots,k_{m-1})$ and Y_2 to $(l_1,\ldots,l_{i-1},l_{i+1},\ldots,l_{m+1})$, and where $k_j < l_i < k_{j+1}$ is satisfied (refer [4], [18-19]). As mentioned before, the coefficients ξ_Y's will always satisfy certain constraints called the Plucker relations. Therefore, for all k_v and l_v, equation (3.1) will give an infinite number of constraints on the ξ_Y's which are the Plucker relations.

The following subsections will show how the Plucker relations can be transformed into the Hirota bilinear form of KP, KdV and Sawada-Kotera equations for their respective τ-

functions, and thus demonstrates the above-mentioned conceptual framework of Hirota-Sato formalism.

3.1 KP Equation

We now wish to show that the τ-function which satisfies the KP equation is of the same form as the Plucker relations, thus verifying the framework.

From equation (2.16), the definition of $\tau(t)$, we may write

$$\tau(t+s) = det\left(\Xi_0^t\, e^{\eta(t,\Lambda)}\, \Xi(s)\right). \tag{3.3}$$

We denote that

$$\Xi(s) = e^{\eta(s,\Lambda)}\, \Xi, \tag{3.4}$$

hence, equation (3.3) can be expressed as

$$\tau(t+s) = \sum_Y S_Y(t)\xi_Y(s), \tag{3.5}$$

where $\xi_Y(s)$ is in the form of equation (2.20) and satisfies all the Plucker relations with the parameters $s = (s_1, s_2, s_3, \ldots)$. Now, applying $S_Y(\tilde{\partial}_t)$ to (3.5) and using the following orthogonality condition, we have

$$S_Y(\tilde{\partial}_t)\,\tau(t+s) = S_Y(\tilde{\partial}_t)\, S_Y(t)\, \xi_Y(s) \tag{3.6}$$

$$S_Y(\tilde{\partial}_t)\,\tau(t+s) = \xi_Y(s). \tag{3.7}$$

We let $t = 0$,

$$\begin{aligned}\xi_Y(s) &= S_Y(\tilde{\partial}_t)\,\tau(t+s)\big|_{t=0}\\ &= S_Y(\tilde{\partial}_s)\,\tau(t+s)\big|_{t=0}\\ &= S_Y(\tilde{\partial}_s)\,\tau(s).\end{aligned} \tag{3.8}$$

Substitute (3.8) into (3.1) yields

$$\sum(-1)^{\delta}\{S_{Y_1}(\tilde{\partial}_t)\tau(t)\}\{S_{Y_2}(\tilde{\partial}_t)\tau(t)\} = 0, \tag{3.9}$$

Expanding equation (3.9) we have,

for $m = 3, j = 1, \delta = 3, i = 1$,

Y_1 corresponding to (k_1, l_1, k_2) whiles Y_2 corresponding to (l_2, l_3, l_4).

For $m = 3, j = 1, \delta = 4, i = 2$,

Y_1 corresponding to (k_1, l_2, k_2) whiles Y_2 corresponding to (l_1, l_3, l_4).

For $m = 3, j = 1, \delta = 5, i = 3$,

Y_1 corresponding to (k_1, l_3, k_2) whiles Y_2 corresponding to (l_1, l_2, l_4).

For $m = 3, j = 1, \delta = 6, i = 4$,

Y_1 corresponding to (k_1, l_4, k_2) whiles Y_2 corresponding to (l_1, l_2, l_3).

Therefore, the corresponding Young diagram of Y_1 and Y_2 of equation (3.9) yields,

$$\begin{aligned}&-(k_1,l_1,k_2)(l_2,l_3,l_4) + (k_1,l_2,k_2)(l_1,l_3,l_4) -\\ &(k_1,l_3,k_2)(l_1,l_2,l_4) + (k_1,l_4,k_2)(l_1,l_2,l_3) = 0\end{aligned} \tag{3.10}$$

We let $(k_1, k_2, l_1, l_2, l_3, l_4) = (0, 1, 2, 3, 4, 0)$,

$$\begin{aligned}&-(0\,1\,2)(0\,3\,4) + (0\,1\,3)(0\,2\,4) - (0\,1\,4)(0\,2\,3)\\ &+ (0\,1\,0)(2\,3\,4) = 0\end{aligned} \tag{3.11}$$

where we have construed equation (3.11) in the fashion of equation (3.9), and thus we have the Young diagram in the form of

$$\begin{aligned}&S_{\emptyset}(\tilde{\partial}_t)\,\tau(t)\; S_{\square\square\\\square\square}(\tilde{\partial}_t)\,\tau(t) - S_{\square}(\tilde{\partial}_t)\,\tau(t) S_{\square\square\\\square}(\tilde{\partial}_t)\,\tau(t)\\ &+ S_{\square\square}(\tilde{\partial}_t)\,\tau(t) S_{\square\\\square}(\tilde{\partial}_t)\,\tau(t) = 0\end{aligned} \tag{3.12}$$

where

$$\tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots\right).$$

By using the definition of $S_Y(t)$ in (2.19), we obtain the value of

$S_{\emptyset} = 1$, $S_{\square} = t_1$, $S_{\square\square} = \frac{t_1^2}{2} + t_2$, $S_{\square\\\square} = \frac{t_1^2}{2} - t_2$, $S_{\square\square\\\square} = \frac{t_1^3}{3} - t_3$ and $S_{\square\square\\\square\square} = \frac{t_1^4}{12} - t_1 t_3 + t_2^2$.

Substituting the value of S_Y into (3.12), yields

$$\begin{aligned}&\tau(t)\left(\frac{1}{12}\frac{\partial^4}{\partial t_1^4} + \frac{1}{2}\frac{1}{2}\frac{\partial^2}{\partial t_2^2} - \frac{1}{3}\frac{\partial^2}{\partial t_1 \partial t_3}\right)\tau(t)\\ &-\frac{\partial}{\partial t_1}\,\tau(t)\left(\frac{1}{3}\frac{\partial^3}{\partial t_1^3} - \frac{1}{3}\frac{\partial}{\partial t_3}\right)\tau(t)\\ &+\left(\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \frac{1}{2}\frac{\partial}{\partial t_2}\right)\tau(t)\left(\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \frac{1}{2}\frac{\partial}{\partial t_2}\right)\tau(t) = 0.\end{aligned}$$

If we simplify further the above equation, then we have

$$\begin{aligned}&\tau(t)\left(\frac{1}{12}\frac{\partial^4}{\partial t_1^4} + \frac{1}{4}\frac{\partial^2}{\partial t_2^2} - \frac{1}{3}\frac{\partial^2}{\partial t_1 \partial t_3}\right)\tau(t)\\ &-\frac{\partial}{\partial t_1}\,\tau(t)\frac{1}{3}\left(\frac{\partial^3}{\partial t_1^3} - \frac{\partial}{\partial t_3}\right)\tau(t)\\ &+\left(\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \frac{1}{2}\frac{\partial}{\partial t_2}\right)\tau(t)\left(\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \frac{1}{2}\frac{\partial}{\partial t_2}\right)\tau(t) = 0.\end{aligned} \tag{3.13}$$

We multiply (3.13) by 24 (so as to obtain the standard form of the KP equation), hence we have

$$\begin{aligned}&2\tau\tau_{t_1^4} + 6\tau\tau_{t_2^2} - 8\tau\tau_{t_1 t_3} - 8\tau_{t_1}\tau_{t_1^3} + 8\tau\tau_{t_1 t_3}\\ &+6\tau_{t_1^2}^2 - 6\tau_{t_1^2}\tau_{t_2} + 6\tau_{t_1^2}\tau_{t_2} - 6\tau_{t_2}^2 = 0\end{aligned} \quad . \tag{3.14}$$

By applying the D-operator properties, we then obtained the KP equation as

$$\left(4D_{t_1}D_{t_3} - D_{t_1}^4 - 3D_{t_2}^2\right)\tau \cdot \tau = 0. \quad (3.15)$$

3.2 KdV Equation

In obtaining the bilinear form of KdV equation, we consider only t_1 and t_3 in the formulation of KP equation, since KdV equation is a one dimensional equation of KP. By substituting the value of S_Y into (3.12), this yields

$$\tau(t)\left(\frac{1}{12}\frac{\partial^4}{\partial t_1^4} - \frac{1}{3}\frac{\partial^2}{\partial t_1 \partial t_3}\right)\tau(t) - \frac{\partial}{\partial t_1}\tau(t)\frac{1}{3}\left(\frac{\partial^3}{\partial t_1^3} -$$
$$\partial\partial t3\ \tau t + 14\partial 2\partial t12\ \tau t\partial 2\partial t12\ \tau t = 0. \quad (3.16)$$

We multiply (3.16) by 24, then we have

$$2\tau\tau_{t_1^4} - 8\tau\tau_{t_1 t_3} - 8\tau_{t_1}\tau_{t_1^3} + 8\tau_{t_1}\tau_{t_3} + 6\tau_{t_1^2}^2 = 0 \ . \quad (3.17)$$

Rearranging the above equation, we obtain

$$2\tau\tau_{t_1^4} - 8\tau_{t_1}\tau_{t_1^3} + 6\tau_{t_1^2}^2 + 4\left(2\tau_{t_1}\tau_{t_3} - 2\tau\tau_{t_1 t_3}\right) = 0. \quad (3.18)$$

Expressing in the D-operator terms, we then obtain KdV equation

$$\left(D_{t_1}^4 - 4D_{t_1}D_{t_3}\right)\tau \cdot \tau = 0. \quad (3.19)$$

3.3 Sawada-Kotera Equation

In obtaining the bilinear form of Sawada-Kotera equation we consider only t_1 and t_5, since the Sawada- Kotera equation is in the fifth order form of KdV equation.

First, we let $(k_1, k_2, l_1, l_2, l_3, l_4) = (0,1,2,3,1,5)$, and substitute this into equation (3.10),

$$-(0\ 1\ 2)(1\ 3\ 5) + (0\ 1\ 3)(1\ 2\ 5) - (0\ 1\ 4)(1\ 2\ 4) +$$
$$(0\ 1\ 5)(1\ 2\ 3) - (0\ 1\ 1)(2\ 3\ 5) = 0 \quad (3.20)$$

where if we interpret equation (3.20) in the structure of equation (3.9), then we obtain the form of the Young diagram as

$S_\emptyset(\tilde{\partial}_t)\,\tau(t)\,S_{\square\square\square\atop\square\square}(\tilde{\partial}_t)\,\tau(t) - S_{\square}(\tilde{\partial}_t)\,\tau(t)\,S_{\square\square\atop\square\square}(\tilde{\partial}_t)\,\tau(t)$
$+S_{\square\square}(\tilde{\partial}_t)\,\tau(t)\,S_{\square\square\atop\square}(\tilde{\partial}_t)\,\tau(t) - S_{\square\square\square}(\tilde{\partial}_t)\,\tau(t)\,S_{\square\atop\square}(\tilde{\partial}_t)\,\tau(t) = 0.$ (3.21)

By using the definition of $S_Y(t)$ in (2.19), we then obtain the following results

$S_\emptyset = 1,\ S_{\square} = t_1,\ S_{\square\square\square} = \frac{t_1^3}{6},\ S_{\square\atop\square} = \frac{t_1^3}{6},\ S_{\square\square\atop\square\square} = \frac{t_1^5}{20} + t_5,\ S_{\square\square} = \frac{1}{2}t_1^2,$

$S_{\square\square\atop\square} = \frac{t_1^6}{45} + t_1 t_5,\ S_{\square\atop\square} = \frac{1}{8}t_1^4.$

Substituting the value of S_Y into (3.21), yields

$$2\,\tau(t)\left(\frac{\partial^6}{\partial t_1^6} + \frac{9}{5}\frac{\partial^2}{\partial t_1 \partial t_5}\right)\tau(t)$$
$$-\frac{\partial}{\partial t_1}\tau(t)\left(12\frac{\partial^5}{\partial t_1^5} + \frac{18}{5}\frac{\partial}{\partial t_5}\right)\tau(t)$$
$$+30\left(\frac{\partial^2}{\partial t_1^2}\right)\tau(t)\left(\frac{\partial^4}{\partial t_1^4}\right)\tau(t)$$
$$-20\left(\frac{\partial^3}{\partial t_1^3}\right)\tau(t)\left(\frac{\partial^3}{\partial t_1^3}\right)\tau(t) = 0.$$

If we rearrange the above equation, then we have

$$2\,\tau(t)\left(\frac{\partial^6}{\partial t_1^6}\right)\tau(t) - 12\left(\frac{\partial}{\partial t_1}\right)\tau(t)\left(\frac{\partial^5}{\partial t_1^5}\right)\tau(t)$$
$$+30\left(\frac{\partial^2}{\partial t_1^2}\right)\tau(t)\left(\frac{\partial^4}{\partial t_1^4}\right)\tau(t) - 20\left(\frac{\partial^3}{\partial t_1^3}\right)\tau(t)\left(\frac{\partial^3}{\partial t_1^3}\right)\tau(t)$$
$$+\frac{18}{5}\tau(t)\left(\frac{\partial^2}{\partial t_1 \partial t_5}\right)\tau(t) - \frac{18}{5}\left(\frac{\partial}{\partial t_1}\right)\tau(t)\left(\frac{\partial}{\partial t_5}\right)\tau(t) = 0. \quad (3.22)$$

Equivalently, we have

$$2\tau\tau_{t_1^6} - 12\tau_{t_1}\tau_{t_1^5} + 30\tau_{t_1^2}\tau_{t_1^4} - 20\tau_{t_1^3}\tau_{t_1^3} +$$
$$18\left(\tau\tau_{t_1 t_5} - \tau_{t_1}\tau_{t_5}\right) = 0 \quad (3.23)$$

By applying the D-operator properties, we then obtain the Sawada-Kotera equation

$$\left(D_{t_1}^6 + D_{t_1}D_{t_5}\right)\tau \cdot \tau = 0. \quad (3.24)$$

4.0 CONCLUSION

The τ-function has acted as a key function to express the solutions of the Sato equation and this is generated from Sato theory. From the nonlinear waves equation being considered, i.e. the KP, KdV and Sawada-Kotera equations, it is shown that the Plucker relations can be represented by the coefficients of τ-function. We then deduce that the τ-function in the bilinear forms of Hirota scheme are closely associated to the Plucker relations in Sato theory. Therefore, we may conclude that the τ-function is essential in indicating this relation between Hirota's direct method and the Plucker relations, and Sato equation in Sato's theory. The above deliberations showed that Hirota's method is linked to Sato theory, and the Hirota-Sato formalism brings to light that the τ-function, which underlies the analytic form of soliton solutions of the related nonlinear waves equations does act as the important function to express the solutions of Sato equation. The Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera equations have been used to corroborate this theoretical framework.

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