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GRAVITY PREDICTION FROM ANOMALY DEGREE VARIANCES

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Abstract

Most of the geodetic boundary value problems were solved in terms of integral formulae extended over the whole earth. These formulae presuppose the knowledge of some of physical quantities at every point on the physical earth surface. In reality these quantities are only measured at a relatively few points on land ,although there are sufficient measurements in large parts of the oceans. Hence prediction of physical quantities is essential to fill in the gaps. The prediction of missing points ând mean anomalies from anomaly degree covariances were investigated. The essential parameters of the covariance functions using anomaly degree variances were computed. The closed form of evaluating empirical covariance functions to compute gravity anomalies with their accompanying statistics were thoroughly discussed.

1.0 INTRODUCTION

Most of the geodetic boundary value problems were solved in terms of integral formulae extended over the whole earth (e.g. Stokes' and, Vening Meinesz'). These formulae presuppose the knowledge of some of the physical quantities e.g. disturbing potential, gravity anomaly, gravity disturbance, geoid undulation, deflections of the vertical and height anomaly at every point on the physical surface of the earth.

On land, even in the densest gravity network, these quantities are only measured at a relatively few points, so that interpolation of these quantities at the other points is inevitable. In some parts of oceans where are no observed physical quantities at all, extrapolation is essential to fill in the gaps. Since mathematically interpolation and extrapolation are the same, we can denote them by the term 'prediction'. Due to the fact that prediction can not yield true values, a best prediction method must be capable ot estimating the interpolation and extrapolation errors of the physical quantities, the effect of these errors in the derived quantities and the effect of neglected distant zones. In this study global and local covariance functions are estimated to allow the prediction of gravity on a global as well as on a local scales.

2.0 GLOBAL GRAVITY ANOMALY COVARIANCE FUNCTIONS

Generally speaking numerical covariance functions for gravity anomalies can be determined from gravity anomaly data. In this investigation, global and local covariance functions are to be estimated from anomaly degree variance models.

2.1 Mathematical models:

The expansion of a global gravity anomaly covariance function between two points P and Q in terms of zonal spherical harmonics is given in [2] as:

$$C(\psi_{PQ})=\sum_{2}^{\infty} C_n P_n(\cos\psi_{PQ}) \tag{1}$$

where:

C_n	:	the conventional gravity anomaly degree variances,
$P_n(\cos\psi_{PQ})$	:	the Legendre polynomial of degree n
ψ_{PQ}	:	is the spherical distance between point P and Q.

The expression for C_n can be given as:

$$C_n=\sum_{o}^{n}\left(\bar{C}_{nm}^2+\bar{S}_{nm}^2\right) \tag{2}$$

where,

$\bar{C}_{nm}$, $\bar{S}_{nm}$ are the fully normalized spherical harmonic potential coefficient sets.

The covariance function $C(\psi_{PQ})$ considered here is assumed to be homogeneous (the function depends only on the relative positions of points P and Q and not on the absolute position of point P) and isotropic (depends on the distance ψ_{PQ} between point P and Q and not on the direction α between them.

From [4], $C(\psi_{PQ})$ is given as:

$$C(\psi_{PQ})=\sum_{o}^{\infty} C_n\left[\frac{R_B^2}{r_P r_Q}\right]^{n+2} P_n(\cos\psi_{PQ}) \tag{3}$$

where,

R_B	:	the radius of the Bjerhammer sphere = 6369.8 km,
r_P	:	the geocentric radius to point P
r_Q	:	the geocentric radius to point Q.

with

$r_P = R + h_P$
$r_Q = R + h_Q$
R being the earth radius = 6371 km.
h_P, h_Q are the ellipsoidal heights at points P and Q.
Taking $r_P = r_Q = R$

$$S=\frac{R_B^2}{r_P r_Q}=\frac{R_B^2}{R^2}$$

we can write eqn (3) as

$$C(\psi_{PQ})=\sum_{o}^{\infty} C_n S^{n+2} P_n(\cos\psi_{PQ}) \tag{4}$$

Equation (4) yields only a point anomaly covariance function. In order to obtain a mean anomaly covariance function $C(\psi_{PQ})$, the averaging operator β_n which is used in physical geodesy to relate spherical harmonics to their mean values over circular areas (spherical cap ψ_o) can be used. Hence the modified version of equation (4) is given in [8] as:

$$\bar{C}(\psi_{PQ})=\sum_{o}^{\infty} \beta_n^2 C_n S^{n+2} P_n(\cos\psi_{PQ}) \tag{5}$$

The recurrence relations for the smoothing function β_n given in [8] as:

$$\begin{aligned} &\beta_o=1 \\ &\beta_1=(1+t_o)/2 \\ &\beta_n=\frac{2n-1}{n+1}\beta_{n-1}-\frac{n-2}{n+1}\beta_{n-2} \end{aligned} \tag{6}$$

with

$$t_o = \cos(\psi_o)$$

Since we usually deal with rectangular blocks, the corresponding circular cap radius ψo can be calculated by equating the areas of the circular cap and the rectangular blocks.

2.2 Anomaly degree variance modelling:

The development of anomaly degree variance model is essential to derive a closed expression for the covariance function of the gravity anomalies and other gravimetric dependant quantities. The basic procedures for this modelling have been discussed in [6]. The modelled anomaly degree variance takes the form:

$$C_n=A\frac{n-1}{(n-2)(n+B)} \tag{7}$$

Best estimates of the parameters A and B are to be found subject to the anomaly degree variances determined from potential coefficients. The anomaly degree variances are computed from [7] as:

$$C_n=\gamma^2(n-1)^2\sum_{o}^{\infty}\left(\bar{C}_{nm}^2+\bar{S}_{nm}^2\right) \tag{8}$$

where,

γ is the normal gravity.

If the degree variances are to be computed using gravity data, the $\bar{C}_{nm}$ and $\bar{S}_{nm}$ coefficient should refer to a level ellipsoid of defined parameters (a, KM,ω,J_2). From eqn (4), the global point anomaly covariance function at ψ = 0 is given as:

$$C(\psi_{PQ}=0)=C_o=\sum_{o}^{\infty} C_n S^{n+2} \tag{9}$$

Where C_o is the global point variance of the anomaly, and it is considered to be representative of the gravity field of the whole earth, unlike the case of regional (local) point anomaly variance whose magnitude depends on the area of interest. The mean point anomaly covariance function is given from eqn (5) as:

$$\bar{C}(\psi_{PQ}=0)=\bar{C}_o=\sum_{o}^{\infty} \beta_n^2 C_n S^{n+2} \tag{10}$$

The summation in the above equation starts from n=0, but in fact we are trying to model the anomaly degree variance C_n from eqn (7) in which summation starts from n = 3 . Thus, we carry out the summation in eqn (10) from n = 3, hence we modify our point and block variances by essentially removing the C_2 value. The accuracy of the gravity anomaly degree variance of eqn. (10) are given in [6] as:

$$V_n^2=0.1058(2n+1)^2 mgals^2 \tag{11}$$

3.0 LOCAL GRAVITY ANOMALY COVARIANCE FUNCTIONS:

3.1 Local covariance functions from gravity data:

For the purpose of prediction of point gravity anomalies in a limited region, a local covariance function could be estimated from the available gravity data. An optimal covariance function is the one which satisfies the following:

* be derived from estimates of the essential parameters in different sample areas.
* be related to a global covariance function in a simple way.
* give consistent results for heterogeneous data groups.
* be easy to compute.

3.1.1. Essential parameters of a local covariance function:

Usually covariance functions are characterized by only a few parameters. For the local gravity anomaly covariance functions there are three essential parameters. These are:

(a) The variance (C_o)

This is the value for the covariance function C(r) for r = 0.

C_o = C(0), r = Rψ

(b) The correlation length (ζ)

This is the value of the argument for which C(r) has decreased to half of it's value at r = 0

$$C(\zeta) = C_o/2$$

(c) The curvature parameter (J)

This is a dimensionless quantity related to the curvature (κ) of the covariance curve at r = 0

$$J = k\ \zeta/C_o$$

The well known formula for the curvature of the curve C(r) gives:

$$k = \frac{C'}{(1+C''^2)^{3/2}} \tag{12}$$

where

$$C' = \frac{dC}{dr}, C'' = \frac{d^2C}{dr^2}$$

for $r = 0$, $C(0) = 0$
and

$$k = C(0)$$

The variance C_o is scale factor for interpolation errors, the curvature parameter J characterizes the behaviour of the covariance function for small distances ψ and the correlation length ζ characterizes the behavior of the covariance function for distances of the order of ζ.

3.1.2 Local covariance function representation:

A local covariance function could be represented by an analytical function of the form given in [3] as:

$$C(r) = C_o\left(1+\frac{r}{d}\right)^{-1} \tag{13}$$

There exist many other alternatives to approximate the above expression analytically. Some of these alternatives are:

1. The power model

$$C(r) = ar^b \tag{14}$$

2. The exponential model

$$C(r) = ae^{-br} \tag{15}$$

3. The logarithmic model

$$C(r) = a + b\log(r) \tag{16}$$

4. The polynomial model

$$C(r) = \sum_{o}^{\infty} a_i r^i \tag{17}$$

Where the coefficients a and b are found by a least squares fit to the empirically determined covariance values.

3.1.3 Local covariance function from anomaly degree variance:

To determine local covariance functions from anomaly degree variances, eqn (10) reduces to:

$$C(\psi_{PQ}) = \sum_{L+1}^{\infty} \beta_n^2 C_n S^{n+2} \tag{18}$$

Where L is the degree of the spherical harmonic model used.

4.0 EMPIRICAL EVALUATION OF THE GLOBAL AND LOCAL COVARIANCE FUNCTION:

The closed form of evaluating empirical global covariance functions are given in [5] as:

$$C(\psi_{PQ}) = \frac{A.S}{B+2}\left[(B+1)\left(F_B - \frac{S}{B} - \frac{S^2 t}{B+1} - \frac{S^3 P_2(t)}{B+2}\right) + F_{-2}\right] \tag{19}$$

$$F_{-2} = \frac{S}{2}(1+3St)M + S^3\left[P_2(t).\log\frac{2}{N} + \frac{1-t^2}{4}\right] \tag{20}$$

with:

M = 1 - L - St

N = 1 + L - St

and FB is found from:

$$F_{n+1} = \frac{1}{n.S}\left[L + (2n-1)tF_n - \frac{(n-1)F_{n-1}}{S}\right]$$

with

$$F_1 = \log\frac{S-t+L}{1-t}$$

$$F_2 = \frac{L - 1 + t\log\frac{S-t+L}{1-t}}{S}$$

$$t = \cos(\psi_{PQ})$$

The constants A is found from eqn (7). The constant B has a fixed value of 24 according to the degree variance model of [6].

The empirical determination of the local gravity anomaly covariance function is obtained from eqn (5) as:

$$C_L(\psi_{PQ}) = C(\psi^1_{PQ}) - \sum_{3}^{L} \beta_n C_n S^{n+2} P_n(\cos\psi_{PQ}) \tag{21}$$

where:

$C_L(\psi_{PQ})$	:	the local empirical covariance function
$C(\psi_{PQ})$	:	the global empirical covariance function as given in eqn. (5).

5.0 LEAST SQUARES PREDICTION:

In predicting gravity anomalies after constructing optimal covariance functions, the prediction equations of [4] are used. Thus using matrix notation:

$$\Delta\hat{g} = C(\bar{C} + \bar{D})^{-1}\Delta g \tag{22}$$

$$\sigma^2_{\Delta g} = C_o - C(\bar{C} + \bar{D})^{-1}C^T \tag{23}$$

where

$\bar{C}$	:	matrix whose elements are the covariance of the known blocks,
C	:	row vector whose elements are variances between the known blocks being predicted and the other blocks used in the prediction,
$\bar{D}$	:	matrix whose diagonal elements are the variances of the known blocks
Δg	:	column vector of the anomalies of the known blocks
C_o	:	variance of the anomaly data
Δg and $\sigma^2_{\Delta g}$	:	predicted gravity anomaly and it's variance.

6.0 COMPUTATIONS:

The data used to compute the covariance functions in this investigation consist of seven different earth models. These models are given in table 1.

The degree variances C_n stemming from the above earth models are computed using eqn (7). The numerical values for the degree variances of the seven models are given in tables 2, 3 and 4.

Table (1) Earth Models considered in the investigation

No.	Earth	Author	Date	Max. Degree
1	GEM9	Lerch et al	1977	20
2	GEM-10B	Lerch et al	1981	36
3	GEM-10N	Lerch et al	1981	22
4	GEM-L2	Lerch et al	1982	20
5	GRIM3L1N	Reigber	1985	36
6	GEM-T1	Marsh et al	1988	36
7	GEM-T2	Lerch et al	1989	50

Table (2): Degree Variances of GEM9, GEM-L2 and GEM-10N Models

n	GEM9 C_n	GEM-L2 C_n	GEM-10N C_n
3	33.53	33.74	33.55
4	21.55	21.68	21.60
5	20.79	20.71	20.69
6	18.98	19.27	19.05
7	19.38	19.77	19.19
8	11.68	11.17	11.42
9	11.46	11.20	11.14
10	10.03	10.09	9.76
11	6.74	6.69	6.62
12	3.65	3.60	3.63
13	6.56	6.77	6.22
14	4.03	3.93	3.43
15	3.29	3.37	3.01
16	2.33	2.53	2.63
17	2.04	2.03	2.17
18	3.30	3.55	3.16
19	2.97	2.99	2.87
20	2.29	2.37	2.01
21	****	****	1.86
22	****	****	1.73

Table (3) : Degree Variances of GEM-10B, GRIM3L1N and GEM-T1

n	GEM-10B C_n	GRIM3L1N C_n	GEM-T1 C_n
3	33.72	33.95	33.73
4	21.75	21.90	21.65
5	20.93	20.16	20.88
6	19.25	19.40	19.72
7	19.35	18.94	19.01
8	11.34	10.60	11.18
9	11.29	10.25	11.25
10	9.57	11.04	9.93
11	6.13	7.89	6.20
12	2.90	4.13	2.52
13	7.58	7.49	7.17
14	3.42	3.67	3.14
15	3.54	3.00	4.40
16	4.18	4.94	3.86
17	3.15	3.82	3.44
18	4.26	3.70	3.77
19	4.27	3.12	2.12
20	3.08	2.71	2.15
21	3.50	2.93	3.49
22	4.18	4.49	2.35
23	2.03	3.05	2.00
24	2.59	2.27	2.07
25	2.96	2.92	1.25
26	2.42	1.76	1.45
27	1.83	1.95	0.81
28	3.21	2.50	1.17
29	2.32	2.35	0.99
30	3.87	2.73	0.97
31	1.75	2.55	0.97
32	1.56	2.25	0.95
33	2.28	2.84	1.35
34	2.77	4.01	0.79
35	3.33	3.61	1.71
36	2.03	3.13	0.77

Table (4) : Degree Variances of GEM-T2 Model

n	C_n	n	C_n	n	C_n
3	33.75	19	2.78	35	1.51
4	21.69	20	2.50	36	0.64
5	20.98	21	3.53	37	0.57
6	19.64	22	2.56	38	0.25
7	19.28	23	2.20	39	0.40

8	11.03	24	2.00	40	0.18
9	11.56	25	1.79	41	0.27
10	9.75	26	1.88	42	0.18
11	6.15	27	1.10	43	0.34
12	2.58	28	1.56	44	0.14
13	7.11	29	1.29	45	0.15
14	3.37	30	1.08	46	0.28
15	3.97	31	1.30	47	0.18
16	3.85	32	1.17	48	0.09
17	3.11	33	1.50	49	0.40
18	3.63	34	1.34	50	0.10

The best estimates of the parameter A in eqn (7) was found from the least squares fit of the computed degree variances of the different earth models. The parameter B is taken as exact integer with a value of 24 as given in [6]. The values for the constant A for different earth models are listed in table (5).

Table (5) : The values of A for different earth models

Earth model	A
GEM9	439.46
GEM-10B	442.95
GEM-10N	439.05
GEM-L2	441.88
GRIM3L1N	442.93
GEM-T1	440.77
GEM-T2	440.99

The global point and mean anomaly variances C_0 and $\bar{C}_0$ are computed for different areas from eqn (10) and eqn (11) respectively. The results are tabulated in tables (6) and (7).

Table (6) : The point and mean anomaly variances of GEM9, GEM-L2 and GEM-10N models in $mgal^2$.

Area	GEM9	GEM-L2	GEM-10N
Point	1840.90	1851.10	1839.17
2' x 2'	1668.52	1677.75	1666.96
5' x 5'	1595.91	1604.74	1594.42
10' x 10'	1556.58	1565.20	1555.11
15' x 15'	1490.52	1498.78	1489.13
20' x 20'	1426.56	1434.46	1425.22
30' x 30'	1319.28	1326.59	1318.04
1^o x 1^o	1096.60	1102.69	1095.57
5^o x 5^o	552.42	555.50	551.89
8^o x 8^o	415.53	417.89	415.13
10^o x 10^o	357.09	359.10	356.75

Table (7) : The point and mean anomaly variances of GEM-10B, GRIM3L1M, GEM-T1 an GEM-T2 models in $mgal^2$.

Area	GEM-10B	GRIM3L1N	GEM-T1	GEM-T2
point	1855.57	1855.50	1846.38	1847.32
2'x2'	1681.82	1681.75	1673.66	1674.33
5'x5'	1608.62	1608.56	1600.66	1601.47
10'x10'	1568.98	1568.92	1561.21	1562.00
15'x15'	1502.40	1502.34	1494.96	1495.72
20'x20'	1437.93	1437.87	1430.81	1431.53
30'x30'	1329.80	1329.75	1323.21	1323.88
$1^o x 1^o$	1105.36	1105.31	1099.87	1100.43
$5^o x 5^o$	556.85	556.83	554.07	554.36
$8^o x 8^o$	418.88	418.86	416.78	416.99
$10^o x 10^o$	359.98	359.97	358.18	358.36

8.0 CONCLUSIONS:

In this study local and global covariance functions were determined from anomaly degree variances as opposed to gravity data. The anomaly degree variances were computed using seven different earth model coefficients sets of table (1). The anomaly degree variances were modelled and the parameter A of eqn. (7) was estimated using least squares criteria.

The global point and mean anomaly variances C_0 and $\bar{C}_0$ were computed and the results are given in tables (6) and (7). The closed form of evaluating empirical covariance functions was outlined. The least squares collocation technique to predict gravity anomalies after the construction of optimal covariance functions is presented.

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