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Taking the Spirit Out of Levelling

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Abstract

Orthometric heights are normally derived using conventional spirit levelling techniques. Such techniques have always been one of the more laborious, time-consuming and costly exercises in surveying. However, with the advent of GPS, ellipsoidal heights can be derived very accurately over a short period. Thus, for many applications, the GPS technique can replace the spirit level method in determining Orthometric height. This is accomplished by converting the GPS derived ellipsoidal height into Orthometric height. This paper describes three method that can be employed for such conversion process. They comprise of a local geoid surface fitting solution, a local gravimetric solution and a global geoid solution. A test network located in the north-east of England was used in this study.

1.0 INTRODUCTION

Traditionally, orthometric height has been derived using conventional spirit levelling techniques. Such techniques have always been one of the more laborious, time-consuming and costly exercises in surveying, and have therefore lead to many new developments to speed up the process (e.g., using reciprocal trigonometrical levelling or by motorised levelling). In a spirit levelling task, one has to proceed setup by setup, from a known height point to station of interest but with GPS one has to occupy only the points of interest and a few additional control points with known height. Thus, for many applications, the determination of orthometric height, normally performed using spirit levelling, is being replaced by the GPS observations method.

The aim of this paper is to describe method used in the estimation of the orthometric height at points where GPS observations are carried out. The three method tested comprises of a local geoid surface fitting solution, a local gravimetric solution and a global geoid solution. A test network located in the north-east of England was used in this study.

2.0 GEOIDAL HEIGHT SOLUTION

2.1 GPS and Geoid-Ellipsoid Separation

The processing of GPS observation is normally done in Cartesian coordinate system, X, Y and Z. The basic results of the precise differential GPS survey of a baseline are the Cartesian oordinate differences X, Y and Z. Baselines connecting the observed GPS points are then put through a network adjustment such as LSD-HEIGHT (Khairul, 1994) or GEOLAB (Bitwise,1991). The resulting X, Y, and Z coordinates of the GPS points are then transformed, using a reference ellipsoid, into geodetic coordinates in terms of latitude ((j>), longitude (A) and ellipsoidal height (h).

GPS ellipsoidal heights are very useful for deformation and subsidence studies and other applications where the emphasis is not so much in locating a precise point in space as in the relative change of height from one time epoch to another. It is, however, the case that the ellipsoidal heights delivered by GPS are not the same as those historically obtained with eodetic levelling which provide orthometric height (H). Conventionally, topographic maps, engineering design and construction project plans, usually depict relief by means of orthometric height. Thus, the application of GPS will be further extended if accurate ransformations between GPS ellipsoidal height differences and the orthometric height ifferences can be realised. This can be accomplished on the condition that we know the geoid height (N) accurately, or rather, the geoid height difference which relates the orthometric height difference to the GPS ellipsoidal height difference (h). The orthometric height (H) is related to the ellipsoidal height (h) by the following relation (see Figure [1]):

$$H = h - N, \text{ or in terms of height differences} \qquad [1]$$

$$\Delta H = \Delta h - \Delta N \qquad [2]$$

The most precise method of obtaining accurate geoid height is using gravimetric observations. Numerical integration of gravimetric observations using Stokes integral equation provides us with a local gravimetric geoid solution. When no local gravimetric geoid solution is available or no gravimetric observations are available as yet, a local geoid surface fitting solution may be employed for small area of gentle undulation, but for large area, then a global geoid solution should be used. However, it is rather important to note that local gravimetric geoid solutions are always based on a global geoid solutions.

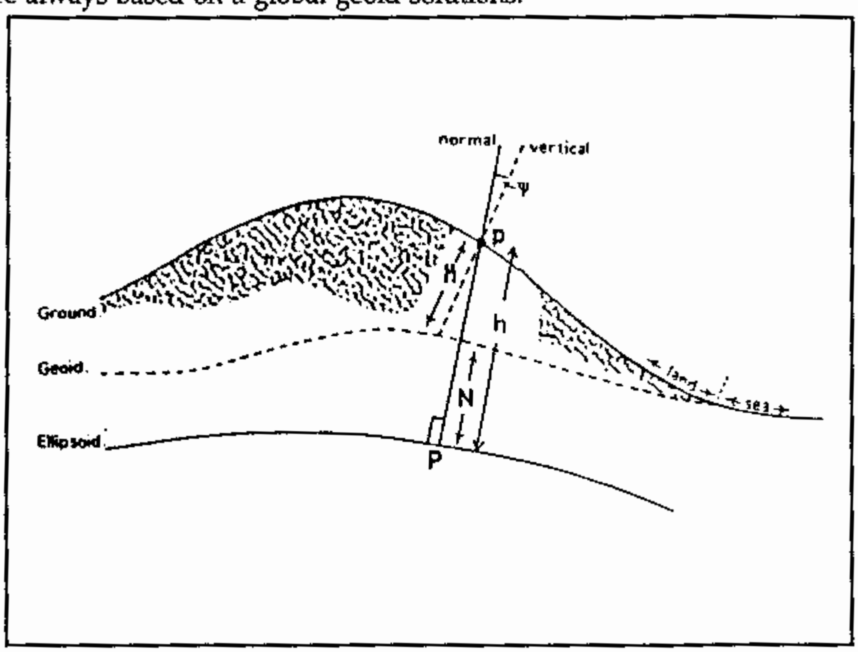

Figure 1 Relationship between Ellipsoidal, Orthometric and Geoid Heights

Global geoid solutions are obtained from global geopotential models which are given as a set of coefficients consisting of a series of spherical harmonic functions. The coefficients of the various terms in the series are determined using a combination of satellite orbit analyses (for the long wavelength geoid features), terrestrial gravity (medium to short wavelength features) and geoid heights measured by satellite altimetry over the ocean (medium to short wavelength features). Some global geopotential models are derived from satellite observations only and are known as satellite-only solutions. These models involve only low-order spherical harmonics and thus contain relatively few coefficients. Table 1 below show examples of such model.

Table 1 Satellite-only Solutions

Global Geopotential Model	Degree (Harmonic)	No. Coefficients
GEM-9 (Lerch et al., 1979)	30	594
GEM-L2 (Lerch et al., 1982)	30	594
GEM-T1 (Marsh et al., 1988)	36	1406
GEM-T2 (Marsh et al., 1989a)	50	2028

Other global geopotential models are obtained by adding surface gravimetry and altimetry data to the satellite-only solutions and they usually contain more coefficients. Table 2 below shows examples of these type of models.

Table 2 Combined Solutions

Global Geopotential Model	Degree (Harmonic)	No. Coefficients
GEM10B (Lerch et al., 1981)	36	1406
RAPP78 (Rapp, 1978)	180	32942
RAPP81 (Rapp, 1981)	180	32942
OSU86F (Rapp and Cruz, 1986)	360	130682
OSU89B (Rapp and Pavlis, 1990)	360	130682
OSU91A (Rapp et al., 1991)	360	130682

The development of the GEM10B, RAPP78 and RAPP81 models are based on the GEM-9 satellite-only model. OSU86F is based on the GEM-L2 model, and OSU89B and OSU91A on the GEM-T2 model.

The geoid height N_{GM} is computed from a set of normalised geopotential coefficients using the following equation:

$$N_{GM} = \frac{GM}{r\gamma}\sum_{n=2}^{n_{MAX}}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}\left[\bar{C}^{*}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right]\bar{P}_{nm}(\sin\phi)$$

... [3]

where,

n_{MAX} is the maximum degree at which the coefficients are known.

$\bar{C}^{*}_{nm}$ are the $\bar{C}_{nm}$ less the zonal coef ficients o f the the normal pot ential of the selected r eference e llipsoid.

G is the gravitational constant.

M is the mass of the earth, including the atmosphere.

a is the earth's equatorial radius.

r is the distance from the earth's centre of mass.

ϕ ,λ are the geocentric latitude and longitude.

$\bar{P}_{nm}(\sin\phi)$ is the normalised associated Lagendre function.

g is the normal gravity.

n, m are the degree and order respectively.

Generally, the more coefficients there are in a model, the more detailed the model usually is since it contains shorter wavelength information of the earth's gravity field. This means that in general, the best solution to use is one that has determined up to the maximum degree and order of 360, which, theoretically at least, can model features in the geoid with half wavelength of 0.5 degrees or 55 km (e.g., OSU86F, OSU89A, OSU91A). This implies that this method is more suitable for areas larger than the stated resolution. In terms of computing time, it can be expected that the higher the degree and order of the harmonic adopted, the more computer time is needed in solving eqn. [3]. A comprehensive discussion and comparison of methods for computing eqn. [3] are given by Tscherning et al. (1983) and Balmino et al. (1991).

Once the geoid heights N_{GM} have been derived, the orthometric height of GPS stations in the network can simply be computed using eqn. [1]. Mainville et al. (1990) found in their study that the very best accuracy of the derived orthometric height that we can expect from this method is around 35 to 40 cm.

In this study, the global geopotential model adopted is the OSU91A which was developed using 30' by 30' mean gravity anomalies derived from terrestrial and altimetric data (Rapp et al., 1991). These data are then combined with GEM-T2 to produce the model complete to degree and order of 360. This model was chosen on the basis that it is the most up-to-date global geopotential model available.

2.3 Local Gravimetric Geoid Solutions

Geoid heights are basically computed using the Stokes integral, integrating gravity anamolies in principle over the surface of the earth. In practice, it is modified to integrate gravity anomalies over a small spherical cap σ_{cap} :

$$N_{grav} = N_{GM} + \frac{R}{4\pi\gamma}\iint_{\sigma_c}(\Delta g - \Delta g_{GM})s(\psi)\,d\sigma$$

... [4]

where;

- N_{grav} is the total geoid height,
- N_{GM} is computed using eqn. [3],
- R is the mean earth radius,
- Δg are the gravimetric observations given as free-air gravity anomalies,
- Δg_{GM} are the free-air gravity anomalies computed from the global geopotential model using an equation similar to eqn.[3], and,
- $s(\psi)$ is Stokes function.

The selection of the 'optimum' cap size is very important. It is preferable to keep the cap size small, since the larger the cap, the longer the computation time. However, as the global geopotential model adopted may have quite large errors in the coefficients of higher degrees, larger cap sizes are required in many cases. For this study, the gravimetric geoid of the British Isles produced by the University of Oxford was used.

The Oxford geoid was computed using the combination of a high degree global potential model and numerical integration of a modified Stokes' integral similar to that described in eqn. [4] (Featherstone, 1992).The global geopotential model used is the OSU91A. The geoid was computed at 2' latitude and 4' longitude grid spacing corresponding to approximately 4 km by 4 km squares over the British Isles. The gravimetric geoid height is computed at the centre of each grid elements to yield 135,000 points comprising geodetic longitude, geodetic latitude and geoid height, all referred to GRS80 (ibid.). This therefore, provides approximately 4 km resolution and the stated accuracy of the geoid height is about 8 cm. To derive the geoid height of a point, the method of Bi-linear interpolation (DMA, 1987) can be used.

2.4 Local Geoid Surface Fitting Solutions

This solution involves the use of a local geoid surface model using a surface fitting procedure. The fundamental theory of this solution is based on the following four assumptions:

(i) the adjusted GPS observations are of very high quality and considered to be exact,
(ii) the orthometric heights of at least three GPS stations in the network are known and also considered to be exact,
(iii) the area involved is small and that the geoid features does not vary rapidly,
(iv) no regional or local gravimetric geoid solution is available for the area due to the non-existence of gravity data.

If we have three or more GPS points with known orthometric heights (referred here as Height Control Point, (HCP), then a local geoid surface solution using a surface fitting model can be employed. This is accomplished by taking N_i as a function of the position of each HCP in the network. The surface fitting model would take the following mathematical form:

$$N_i = F(\phi_i, \lambda_i) = a_0 + a_1x_i + a_2y_i + a_3x_iy_i + \dots\dots\dots \quad [5]$$

where;

a_0, a_1, a_2, a_3 are the unknown model coefficients,
$x_i = (\phi_i - \phi_o)\ \rho_m/\mu$
$y_i = (\lambda_i - \lambda_o)\ \nu_m \cos\phi_m/\mu$
$\phi_m = (\phi_i + \phi_o)/2$

and;

ϕ_i, λ_i are the latitude and longitude of station i,
ϕ_0, λ_0 are the latitude and longitude of a point chosen as the origin,
ρ_m is the prime vertical radius of curvature at mid-latitude,
ν_m is the meridian radius of curvature at mid-latitude,
ϕ_m is the mid-latitude between ϕ_i and ϕ_o, and,
μ = 206264.806"

When using 3 HCPs, eqn. [5] will represent a simple plane surface passing through all the three points on the surface of the geoid. The east-west and the north-south tilt of the plane surface are designated by the model coefficients a_o, a_1 and a_2. Once these coefficients are computed using the least squares method, the unknown orthometric height of the other GPS points can then be derived. If more than three HCPs are available, a more complex (curve) surface can be modelled in place of the plane surface.

In this paper, two surface fitting models are presented and they take the following form:

$$\text{Model P1}: \quad N_i = a_0 + a_1x_i + a_2y_i \quad [6]$$

$$\text{Model P2}: \quad N_i = a_0 + a_1x_i + a_2y_i + a_3\Delta h_i \quad [7]$$

where $\Delta h_i = h_i - h_0$, h_i being the ellipsoidal height of the GPS point and h_0 is the ellipsoidal height at the point selected as the origin. The rational of adding the height term h in the model P2 is based on the assumption that the geoid undulation follows approximately that of the topography.

3.0 TEST NETWORK

3.1 Windmill Hill Network Experiments

The site chosen for the orthometric height experiments is the Windmill Hill area located in the north-east of England. This site has considerable variation in topography and landscape, with an overall east-west drop in height of about 200 metres. This network was originally planned to consist of points located in a grid form at 300 metres apart, but due to difficult terrain, the final grid has become slightly irregular and also appears to be incomplete. The Windwill Hill network consist of thirty-four GPS points with known orthometric heights and covers an area of approximately 2.5 km by 3 km as illustrated in Figure [2].

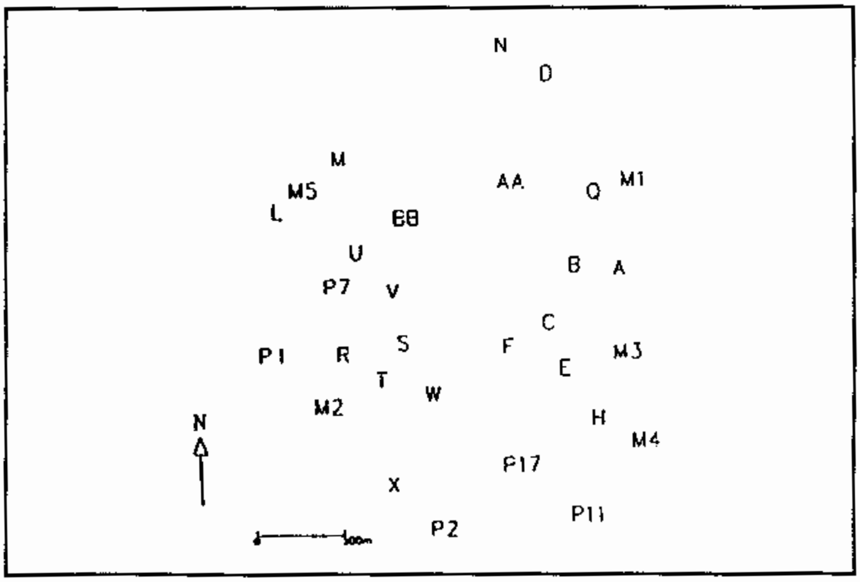

Figure 2 Windmill Hill Network configuration.

3.1.1 Observations and Processing

GPS observations was carried out at thirty-four points in the network. Four units of the portable Ashtech™ receivers were employed in the observation scheme. Observation period adopted at each GPS point was about 90 minutes. Special attention was paid to centring the antennas over the observed points and the measurements of the antenna height, i.e, the height between the edge of the ground plane of the antenna and the ground point. This ensures the resulting ellipsoidal height will be as accurate as practically possible. The baselines together with their associated quality measures in the form of covariance matrix were derived using the Ashtech™ post-processing software packages (Ashtech, 1991) using raw data measured by the Ashtech™ receivers.

The orthometric heights of the thirty-four points were determined by conventional spirit levelling and referred to the Ordnance Survey benchmarks in the area.

3.2 Orthometric Height Experiments

The network estimation program L3D-HEIGHT, was used for the simultaneous three-dimensional coordinates and orthometric height estimation. A total of 81 baselines were processed using three fixed stations.

For each of the two surface fitting models, three tests were performed using three different subsets of HCPs. The results of the three tests are given in Table [3] below. The three tests indicate that the two model produces similar accuracy in the orthometric height estimation with r.m.s of less that 1 cm. Each test uses a different set of HCPs with a different distribution pattern. It can be seen from the result that the distribution pattern does not affect very much the overall result. Adding a height term in the model does not gives an improvement to the results as anticipated. Figure [3] show a scatter plot of the differences between known orthometric height differences and derived height differences (from model P2) computed at one fixed point. It can be seen that most of the computed height differences in this network are within the allowable tertiary misclosure. This seems to suggest that for relatively flat area, the surface fitting models can be use as an alternative to spirit levelling in the determination of orthometric heights.

Table 3 Summary of r.m.s of height differences computed from surface fitting models and known orthometric height in the Windmill Hill network (in cm.)

Geoid Model	No. HCPs	Test No.1	Test No.2	Test No.3
P1	3	0.6	0.8	0.8
P2	4	0.6	0.5	0.6

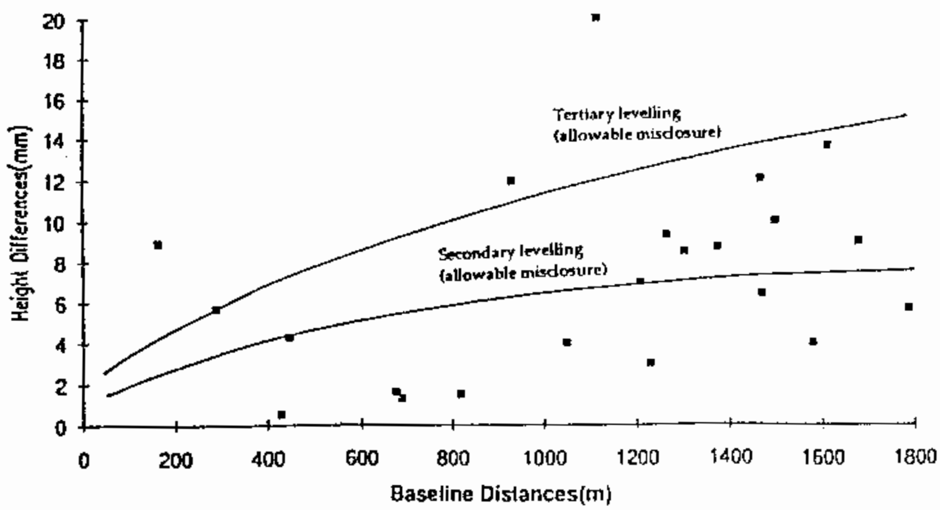

Figure 3 Scatter plot of the difference between known orthometric height difference and the model P2 derived height difference

The OSU91A model was also used to derive orthometric heights. Table [4] give the results obtained from the solution. The results show that using geoid heights computed directly from the OSU91A model, an estimated accuracy of about 30 cm can be expected. Here, we can safely conclude that for a relatively flat area, where three or more points with known orthometric heights are available, it is much simpler and more economical to use a local geoid surface fitting solution.

Table 4 Summary of r.m.s. of differences in height computed between the OSU91A model, OXFORD model and known orthometric height at the Windmill Hill network (in cm.)

Geoid Model	No. HCPs	Test No.1	Test No.2
OSU91A	-	26.1	26.3
OXFORD.	-	14.6	14.2

Table [4] also shows the results derived using the OXFORD model. It can be seen that the accuracy given by the OXFORD model is about 50% better than OSU91A. Figure [4] shows that using the OXFORD model over a relatively flat area will give an accuracy comparable to that given by tertiary levelling can be achieved.

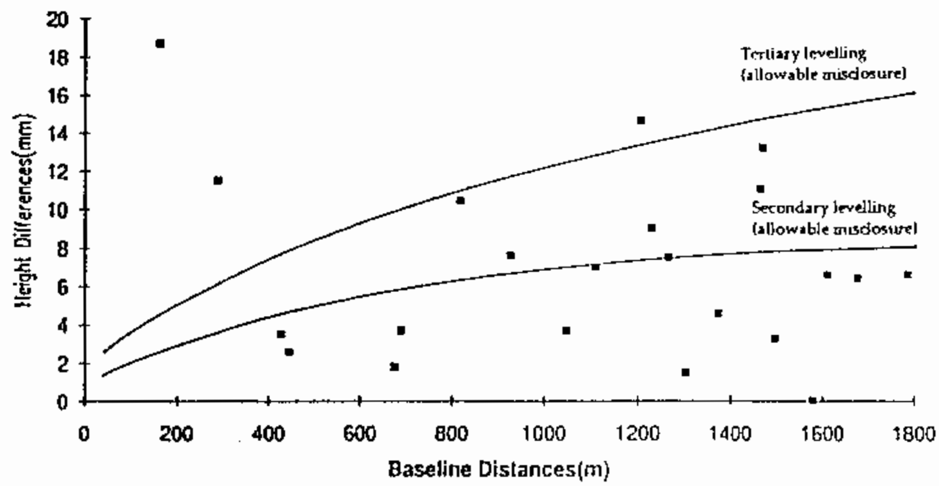

Figure 4 Scatter plot of the difference computed between the known orthometric height difference and the OXFORD model derived height difference

4.0 CONCLUDING REMARKS

It has been proven that GPS observables can be used to estimate orthometric heights. Thus, for some applications, this has the potential for replacing conventional spirit levelling for height determination.

In small areas, typically the size of a few kilometres square, with smooth geoid features, any of the surface fitting models tested may be used for orthometric height estimation since they have shown to provide similar accuracy (at about 1.0 cm level). The accuracy that may obtained using local geoid surface fitting solutions is at least comparable to that given by tertiary levelling. This has been demonstrated by the experiments performed on the Windmill Hill network. However, for practicality, a simple plane model is much preferred than those which are more complicated, on the basis that it needs the least number of points with known orthometric height for computing the model coefficients. It has also been shown that taking into account the topography of the area of interest by representing the height of terrain as a simple linear term in the surface model does not seem to contribute any significant improvements to the orthometric height estimation.

Using a global geoid solution or a local gravimetric geoid solution alone to estimate the orthometric height in small area (of a few kilometres square) has shown to provide accuracies of about 15 - 30 cm which is significantly worse (even for areas with smooth geoid) than those obtained by local geoid surface fitting solutions. This implies that for small areas with a relatively smooth geoid, it is much simpler and economical to use a local geoid surface fitting solution to estimate the orthometric height. The experiments also indicate that the accuracy obtained from a local gravimetric geoid solution is normally about 50% better than those obtained from a global geoid solution. This improvements is contributed by the presence of some of the short wavelength features of the local geoid in the local gravimetric geoid model. Thus, whenever possible, a local gravimetric geoid solution should be used instead of a global geoid solution.

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