THE HOMOLOGICAL FUNCTORS OF A BIEBERBACH GROUP OF DIMENSION FOUR WITH DIHEDRAL POINT GROUP OF ORDER EIGHT

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To the endless list of people who are meant very much to me

I address you alphabetically because there is no order on how I need you

I love you all

The one who inspire me and give me strength

My beloved mother,

Pn Siti Rugayah binti Hj.Sirah

My late father,

En.Mohammad bin Uyop

My siblings,

Mohd Fikri, Mohd Fahmi, Siti Amanina, Siti Nazihah, Muhammad Syukri

Lecturers and friends

Thank you very much.

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ABSTRACT

A Bieberbach group is a torsion free crystallographic group. It is an extension of a free abelian group of finite rank by a finite point group. The homological functors are originated in homotopy theory. In this research, some homological functors such as $J(G)$, $\nabla(G)$ and the exterior square of a Bieberbach group of dimension four with dihedral group of order eight are computed. This research is an extension to the research on finding the nonabelian tensor square of the same group. One of the methods to find the homological functors of the group is to use its nonabelian tensor square of the group. Therefore, the result of the nonabelian tensor square of the Bieberbach group of dimension four with dihedral point group of order eight is used to compute its homological functors. A software named Groups, Algorithms and Programming (GAP) is also used to identify some homological functors of a Bieberbach group of dimension four with dihedral group of order eight.

ABSTRAK

Kumpulan Bieberbach merupakan kumpulan kristalografi yang bebas kilasan. Ia adalah perluasan daripada kumpulan abelan bebas peringkat terhingga melalui kumpulan titik terhingga. Fungtor homologi berasal daripada teori homotopi. Di dalam kajian ini, beberapa fungtor homologi seperti $J(G), \nabla(G)$ dan kuasa dua peluaran untuk kumpulan Bieberbach yang berdimensi empat dengan kumpulan titik dwihedron berperingkat lapan telah dihitung. Kajian ini merupakan lanjutan kepada kajian yang tertumpu kepada kuasa dua tensor tidak abelan terhadap kumpulan yang sama. Salah satu cara untuk menghitung fungtor homologi sesuatu kumpulan ialah dengan menggunakan kuasa dua tensor tidak abelannya. Oleh itu, hasil kajian terhadap kuasa dua tensor tidak abelan kumpulan Bieberbach yang berdimensi empat dengan kumpulan titik dwihedron telah digunakan untuk menghitung fungtor homologinya. Perisian "*Groups, Algorithms and Programming*" (GAP) juga telah digunakan untuk mengenalpasti beberapa fungtor homologi untuk kumpulan Bieberbach yang berdimensi empat dengan kumpulan titik dwihedron berperingkat lapan.

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LIST OF SYMBOLS

1	Identity element
C_n	Cyclic group of order n
GAP	Groups, Algorithms and Programming
G	A group G
$\lvert G \rvert$	Order of the group G
$G \otimes G$	Tensor square of G
$G \tilde{\otimes} G$	Symmetric square of G
g^h	The conjugate of g by h
G'	The commutator subgroup of G
G/G'	The abelianization of G
$G \wedge G$	Exterior square of G
$\ker(\kappa)$	Kernel of the homomorphism κ
$M(G)$	Schur multiplier of G
$[x, y]$	The commutator of x and y
$\subseteq$	Subset of
$\in$	Element of
$\wedge$	Wedge product
$\times$	Direct product
$>$	Greater than
$<$	Less than
$\geq$	Greater than or equal
$\leq$	Less than or equal

CHAPTER 1

INTRODUCTION

1.1   Introduction

The main focus in the development of this research is the homological functors of a Bieberbach group of dimension four with dihedral point group of order eight. Algebraic K-theory and homotopy theory are the origins of the nonabelian tensor square and homological functors of a group.

The homological functors of a group G such as $J(G)$, $\nabla(G)$, the exterior square, the Schur multiplier, $\Delta(G)$, the symmetric square and $\tilde{J}(G)$ are closely related to the nonabelian tensor square of the group. The nonabelian tensor square $G \otimes G$ is a special case of the nonabelian tensor product where the product is defined if the two groups act on each other in a compatible way and their actions are taken to be conjugation. The nonabelian tensor square $G \otimes G$ is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations

$$gh \otimes k = (g^h \otimes k^h)(h \otimes k) \text{ and } g \otimes hk = (g \otimes k)(g^k \otimes h^k)$$

for all $g, h, k \in G$ where $g^h = h^{-1}gh$.

A Bieberbach group is defined to be a torsion free crystallographic group. Meanwhile, a crystallographic group is a discrete subgroup G of the set of Euclidean space $\mathbb{E}^n$, where the quotient space $\mathbb{E}^n/G$ is compact. A Bieberbach group is an extension of a free abelian group, L of finite rank by a finite group, P. In other words, there is a short exact sequence $1 \to L \xrightarrow{\varphi} G \xrightarrow{\phi} P \to 1$ such that $G/\varphi(L) \cong P$. The group G is called the Bieberbach group with point group P. The dimension of the Bieberbach group is the rank of L.

In this research the computations of some homological functors of one of 73 Bieberbach groups of dimension four with dihedral point group of order eight, G_B are presented. Later on, G_B is used to denote a Bieberbach group of dimension four with dihedral point group of order eight. The result of the nonabelian tensor square of the same group is used to compute its homological functors. Furthermore, in this research the algebraic computations of some homological functors of G_B are also computed by using Groups, Algorithms and Programming (GAP) software.

1.2   Research Background

This research is an extension to the research in [1] on finding the nonabelian tensor squares of G_B. In [1], Mohd Idrus found 73 Bieberbach groups with dihedral point group of order eight using the Crystallographic Algorithms and Tables (CARAT) package. The results of the nonabelian tensor squares found in [1] are used in the computations of the homological functors of G_B. The homological functors under consideration are including $J(G_B), \nabla(G_B)$ and the exterior square.

1.3   Problem Statement

What are $J(G_B), \nabla(G_B)$ and the exterior square for a Bieberbach group of dimension four with dihedral point group of order eight?

1.4   Research Objectives

The main objectives of this research are:

1. To explore the various concepts and properties of G_B and their nonabelian tensor squares.

2. To determine some homological functors such as $J(G_B), \nabla(G_B)$ and the exterior square of G_B by using hand computations.

3. To compute some homological functors such as $J(G_B), \nabla(G_B)$ and the exterior square of G_B by algebraic computation using GAP software.

1.5   Scope of the Study

This research focuses on one of 73 Bieberbach group of dimension four with dihedral point group of order eight, G_B listed in CARAT package. The homological functors which only include in this research are $J(G_B), \nabla(G_B)$ and the exterior square of G_B.

1.6 Significance of the Study

Mathematical approach are now widely been used in many problems including the problems involving the pattern of crystals. This group is used in the classification of the crystals. New constructions of the crystallographic groups including the Bieberbach groups might give some interest to the chemists and physicist. Such constructions are the nonabelian tensor square and homological functors of a group.

Furthermore, this research will lead to the development of new theorems with proofs. The results of this research can be presented in a conference and can be sent for publication in a journal. Moreover, the results of this research can be used for further research that related to the problem on finding the homological functors of Bieberbach groups of any dimension with any point group.

1.7 Research Framework

This research has been carried out according to the following steps:

1. Study the various concepts and properties for a Bieberbach group of dimension four with dihedral point group of order eight

2. Prove in details of its polycyclic presentation with the help of GAP and study its nonabelian tensor square.

3. Study the related results that are used to compute the homological functors for a Bieberbach group with certain dimension with certain point group.

4. Determine and compute some homological functors such as $J\left(G_B\right), \nabla\left(G_B\right)$ and the exterior square of a Bieberbach group of dimension four with dihedral point group of order eight by hand calculation and GAP software.

5. Dissertation writes up.

6. Presentation of dissertation.

1.8 Dissertation Report Organization

This dissertation is organized into five chapters. Chapter 1 is the introduction chapter. This chapter includes the research background, problem statement, research objectives, scope of the research, significance of study and the research framework.

In Chapter 2, some definitions and basic concepts that are used throughout the dissertation are included. Besides, some previous researches on the nonabelian tensor squares and homological functors of some groups are included. Furthermore, some researches on Bieberbach groups are also stated. In addition, this chapter also includes some descriptions on Groups, Algorithms and Programming (GAP) software together with Crystallographic Algorithms and Tables (CARAT) package.

Chapter 3 begins by reviewing on some preliminary results found in previous research on the nonabelian tensor square of a Bieberbach group of dimension four with dihedral point group of order eight, G_B. It is shown in previous research that the matrix presentation of G_B listed in CARAT package can be transformed into polycyclic presentation. Hence, in this chapter the transformation of the group, G_B into polycyclic presentation is proved in details and it is proven that the polycyclic presentation shown by GAP is the same as polycyclic presentation shown in [1].

Then, from its polycyclic presentation the nonabelian tensor square of G_B are determined by Mohd Idrus in [1].

Chapter 4 discusses on new results of the homological functors such as $J(G_B), \nabla(G_B)$ and the exterior square of a Bieberbach group of dimension four with dihedral point group of order eight that are done manually using related theorems and definitions and algebraic computation by GAP software.

Lastly in Chapter 5, the results obtained are summarized. Suggestions for future research are also included.

1.9   Conclusion

This chapter provides the research background, problem statement, research objectives, scope of the research, significance of the study, the research framework and dissertation report organization. In the next chapter, literature reviews of this research were discussed.

REFERENCES

1. Mohd Idrus, N. *Bieberbach Groups with Finite Point Groups*. Ph.D. Thesis. Universiti Teknologi Malaysia; 2011.

2. Brown, R. and Loday, J.-L. Van Kampen Theorems for Diagrams of Spaces. *Topology*. 1987. 26: 311-335.

3. Brown, R., Johnson, D. L. and Robertson, E. F. Some Computations of Nonabelian Tensor Products of Groups. *Journal of Algebra*. 1987. 111:177-202.

4. Bacon, M. and Kappe, L.C. On Capable *p*-Group of Nilpotency Class Two. *Illinois Journal of Math.* 2003. 47:49-62.

5. Rotman, J.J. *An Introduction to the Theory of Groups.* 4th. ed. New York: Springer-Verlag. 1995.

6. Eick, B. and Nickel, W., Computing Schur Multiplicator and Tensor Square of Polycyclic Group. *Journal of Algebra*, 2008. 320(2):927-944.

7. Rocco, N.R., On a Construction related to the Nonabelian Tensor Square of a Group. *Bol.Soc.Brasil.Mat. (N.S).* 1991. 22(1):63-79.

8. Blyth, R. D. and Morse, R. F. Computing the Nonabelian Tensor Squares of Polycyclic Groups. *Journal of Algebra.* 2009. 321:2139-2148.

9. Ellis, G. and Leonard, F. Computing Schur Multipliers and Tensor Products of Finite Groups. *Proc.Roy.Irish Acad, Sect. A*. 1995. 95(2):137-147.

10. Blyth, R.D., Fumagalli, F. And Morigi, M. Some Structural Results on the Non-abelian Tensor Square of Groups. *Journal of Group Theory*. 2010. 13:83-94.

11. Kappe, L.C., Sarmin, N.H., and Visscher, M.P. 2-generator 2-groups of Class 2 and their Nonabelian Tensor Squares. *Glasgow Math. J.* 1999. 41:417-430.

12. Mohd Ali, N.M., Sarmin, N.H. and Kappe, L.C. Some Homological Functors of Infinite Non-abelian 2-generators Groups of Nilpotency Class 2. *Proceeding of 15ᵗʰ Mathematical Sciences National Conference (SKSM 15).* 2007. 71-76.

13. Ramachandran V.A.R., Sarmin, N.H. and Mohd Ali, N.M. The Nonabelian Tensor Square and Homological Functors of S_3. *Proceeding of Regional Annual Fundamental Science Seminar.* 2008. 350-354.

14. Mat Hassim, H.I., Sarmin, N.H., Mohd Ali, N.M. and Mohamad, M.S. On the Computations of Some Homological Functors of 2-Engel Groups of Order at Most 16. *Journal of Quality Measurement of Analysis.* 2011. 7(1):153-159.

15. Hiller, H. Crystallography and Cohomology of Groups. *The American Mathematical Monthly.* 1986. 93 (10): 765-779.

16. Plesken, W. And Schulz, T. Counting Crystallographic Groups in Low Dimensions. *Experimental Mathematics*. 2000. 9 (3): 407-411.

17. Bovdi, V. A., Gudivok, P. M. and Rudko, V. P. Torsion Free Groups with Indecomposable Holonomy Group I. *Journal of Group Theory*. 2002. 5:75-96.

18. Bovdi, V. A., Gudivok, P. M. and Rudko, V. P. Torsion Free Crystallographic Groups with Indecomposable Holonomy Group II. *Journal of Group Theory*. 2004. 7:555-569.

19. Cid, C. And Schulz, T. Computation of Five and Six Dimensional Bieberbach Groups. *Experimental Mathematics*. 2001. 10 (1): 109-115.

20. Russo, F. G., Erfanian, A. and Sarmin, N. H. The Nonabelian Tensor Product of Crystallographic Groups and Related Questions. *Asian European J.Math*. 2011. 4:271-282.

21. Masri, R. *The Nonabelian Tensor Squares of Certain Bieberbach Groups with Cyclic Point Group of Order Two*. Ph.D. Thesis. Universiti Teknologi Malaysia; 2009.

22. Mat Hassim, H.I., Sarmin, N.H., Mohd Ali, N.M., Masri, R. and Mohd Idrus, N. The Schur Multipliers of Certain Bieberbach Groups with Abelian Point Groups. *Proceeding of the 20th National Symposium on Mathematical Sciences* (*SKSM 20), AIP Conf. Proc.* 2013. 1522:1069-1074.

23. Mat Hassim, H.I., Sarmin, N.H., Mohd Ali, N.M., Masri, R. and Mohd Idrus, N. On the Generalization of a Homological Functor of Bieberbach Group with Cyclic Point Group. *Proceeding of 21st National Symposium on Mathematical Sciences* (*SKSM 21*), *AIP Conf. Proc.* (*In press*). 2013.

24. The GAP Group. GAP-Groups, Algorithms and Programming. Version 4.4.10: 2007 (http://www.gap-system.org).

25. Torsion Free Space Groups. (http://wwwb.math.rwth-aachen.de/carat/bieberbach.html).