

THE COMMUTATIVITY DEGREE OF SOME NONABELIAN TWO-GROUPS
WITH A CYCLIC SUBGROUP OF INDEX FOUR

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Specially dedicated to my beloved father, mother, my wife and all my family members. Thank you to Dr Nor Muhainiah Mohd Ali, my friends and all those people who have guided and supported me throughout my journey in education.

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ABSTRACT

The determination of the abelianness of a finite group has been introduced for symmetric groups, finite groups and finite rings in the last fifty years. The abelianness of a group or known as the commutativity degree of a group is defined as the probability that a random pair of elements in the group commute. The basic probability theory is used in studying its connection with group theory. The aim of this study is to determine the commutativity degree of some nonabelian 2-groups with a cyclic subgroup of index 4. Two approaches have been used to calculate the commutativity degree of those groups. The first approach is by using a formula involving the number of conjugacy classes and the second approach is by using the Cayley Table method. In this thesis, some basic concepts of the commutativity degree of finite group are first reviewed then the computation of the commutativity degree of nonabelian 2-groups with a cyclic subgroup of index 4 of order 16 and 32 are done.

ABSTRAK

Penentuan keabelanan kumpulan terhingga telah diperkenalkan bagi kumpulan simetri, kumpulan terhingga dan gelanggang terhingga dalam lima puluh tahun yang lalu. Keabelanan sesuatu kumpulan atau dikenali sebagai kekalisan tukar tertib sesuatu kumpulan ditakrifkan sebagai kebarangkalian sepasang unsur-unsur rawak dalam kumpulan tersebut berkalis tukar tertib. Asas teori kebarangkalian digunakan dalam mengkaji hubungannya dengan kumpulan teori. Tujuan kajian ini adalah untuk menentukan darjah kekalisan tukar tertib bagi beberapa kumpulan-2 tak abelian dengan satu subkumpulan kitaran berindeks 4. Dua pendekatan telah digunakan untuk menghitung darjah kekalisan tukar tertib kumpulan-kumpulan tersebut. Pendekatan pertama ialah dengan menggunakan formula yang melibatkan bilangan kelas konjugat dan pendekatan kedua ialah dengan menggunakan kaedah Jadual Cayley. Dalam tesis ini, beberapa konsep asas mengenai darjah kekalisan tukar tertib kumpulan terhingga pada mulanya dikaji semula, kemudian pengiraan darjah kekalisan tukar tertib bagi kumpulan-2 tak abelian dengan subkumpulan kitaran berindeks 4 peringkat 16 dan 32 diselesaikan.

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LIST OF SYMBOL/ ABBREVIATIONS/ NOTATION

e	Identity element
$P(G)$	Commutativity Degree
$k(G)$	Number of Conjugacy classes
G	A finite group G
$ G $	Order of a finite group G
$cl(a)$	The Conjugacy class of a in G
x^{-1}	Inverse of x
\in	Element of
$=$	Equal to
\neq	Not equal to
\forall	For all
\times	Direct product
$\langle a \rangle$	Subgroup generated by a
\geq	Greater than or equal to

CHAPTER 1

INTRODUCTION

1.1 Introduction

In the past years, the probabilistic theory was proved to be beneficial in the solution of different complicated problems in group theory. The question that come arised from the previous researches was “Can someone determine the abelianness for a nonabelian group?” Let G be a group and let x, y be elements in G . We consider the total number of pairs of (x, y) for which x and y commute (i.e $xy = yx$) and then divide it by the total number of pairs of (x, y) which is possible. Thus, the result will give the probability that a random pair of element in a group commutes or the commutativity degree of a group G .

Finding the commutativity degree of a finite group is equivalent to finding the number of conjugacy classes of the group. Besides that, the results of finding the commutativity degree can be applied in many areas of group theories. Moreover, there are many questions, and a long history of research of finding the commutativity degree in the literature.

In this research, G is always considered as a finite group. The commutativity degrees of some finite groups have been determined by many researchers including Castelaz in [1].

1.2 Research Background

The idea of finding $P(G)$, which is the commutativity degree of a group G , was introduced by Erdos and Turan [2] in which they studied the properties of random permutations, and develop a statistical theory for the symmetric group. Gustafson [3] and MacHale [4] have completed the research about the concept of finite groups and they found that the commutativity degree of some finite groups. Furthermore, the commutativity degree of group G is mathematically defined as follows:

$$P(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}, \text{ where } |G| \text{ is the order of group } G$$

In [3], Gustafson introduced the formula to find the commutativity degree as $P(G) = \frac{k(G)}{|G|}$ where $k(G)$ is the number of conjugacy classes of G . This formula is the most important relation between the commutativity degree and conjugacy classes of finite group G .

Several authors have done the researches of this probability theory for specific groups, including Balcastro and Sherman [5] for counting centralizer in finite groups, MacHale [4] for finite groups and finite ring, Sarmin and Mohamad [6] for 2-generator 2-groups of nilpotency class 2, Rusin [7] for finite groups, Abdul Hamid [8] for Dihedral groups and Abu Sama [9] for alternating groups.

Further study was then done on this topic by Castelaz [1] in 2010, as she worked on solvable and non-solvable groups where she provided two upper bound on the commutativity degree of non-solvable groups. Furthermore, she described all groups with commutativity degree in term of solvability of the groups.

1.3 Statement of the Problem

This research specifically studies the commutativity degree of nonabelian groups of order 16 of index 4 and some nonabelian groups of order 32 with cyclic subgroup of index four.

1.4 Research Objectives

The objectives of this study are:

- i. to study and understand the concepts and properties of nonabelian 2-groups with a cyclic subgroup of index 4.
- ii. to study the commutativity degree of some finite groups G .
- iii. to find the number of conjugacy classes of nonabelian 2-groups of order 16 with a cyclic subgroup of index 4.
- iv. to find the commutativity degree of nonabelian groups of order 16 and of some nonabelian groups of order 32 with cyclic subgroup of index 4.

1.5 Significance of the Study

The study of commutativity degree has been widely known, that led many researchers to do more a scientific on this study. The aim of this study is to provide some accurate research in group theory. Moreover, these results of commutativity degree can be transferred to non-commuting graph. It can also be applied in the field of graph theory to model many types of relations and process dynamics in physical, biological and social systems.

1.6 Scope of the Study

This research focuses only on the commutativity degree of group for nonabelian groups of order 16 and some nonabelian groups of order 32 with a cyclic subgroup of index 4.

1.7 Research Methodology

This dissertation is carried out according to the following steps:

1. Literature study and overview of some finite 2-groups together with their number of conjugacy classes and commutativity degree.
2. Literature study of the classification of finite nonabelian 2-groups with a cyclic subgroup of index 4.
3. Finding the number of conjugacy class of some finite nonabelian 2-groups with a cyclic subgroup of index 4.

4. Finding the commutativity degree of some nonabelian 2-groups with a cyclic subgroups of index 4
5. MSc dissertation writing, submission and presentation.

1.8 Dissertation Report Organization

This dissertation is organized into five chapters. Chapter 1 includes the introduction of the research, research background, statement of the problem, research objectives, significance and scope of the study.

In Chapter 2, the history of the commutativity degree of finite groups is overviewed and some definitions, basic concepts and theorem that are used throughout the dissertation are introduced. Moreover, the formula to find the commutativity degree using Cayley Table and the number of conjugacy classes are stated in this chapter. Besides, some results on the commutativity degree of some finite groups from previous researchers are also stated.

In Chapter 3, the commutativity degree of nonabelian groups of order 16 with a cyclic subgroup of index 4 is determined by using two different methods. The methods are the number of conjugacy classes method and Cayley Table method. Meanwhile, in Chapter 4, the commutativity degree of some nonabelian groups of order 32 with a cyclic subgroup of index 4 is determined by using Cayley Table method.

Finally, Chapter 5 concludes the research, and presents recommendations and suggestions for future work.

1.9 Conclusion

This chapter serves as the introduction of this MSc dissertation. The research background, statement of the problem, research objectives, significance of the study, scope of research methodology and dissertation report organization are included in this chapter.

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