# THE APPLICATION OF FINITE ELEMENT METHOD IN 2D HEAT DISTRIBUTION PROBLEMS FOR IRREGULAR GEOMETRY

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*To my beloved father and mother,*

*Ahmad Kailani bin Kosnin*

*Re'ha binti Junit*

*To my supervisor,*

*Dr. Yeak Su Hoe*

*And also to all my friends.*

*Thank you for your love, support and guidance.*

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# ABSTRACT

In mathematics, the finite element method (FEM) is a numerical technique for finding approximate solutions of boundary value problems from differential equations. The term 'finite element' stems from the procedure in which a structure is divided into small but finite size elements. FEM is very useful for problems with complicated geometries, loadings, and material properties where analytical solutions cannot be obtained. In this research, simple irregular problem is used as an example of industry problems to be solved using FEM and finite difference method (FDM). Matlab programming is used as a calculation medium for both FEM and FDM methods respectively. Since the results of the problem for both methods converge, it also proves that the results are valid. Hence we can conclude that simple irregular problem can be solved using FEM and FDM. From this research, we also discovered that FEM produces more stable and consistent result compared to FDM for the solution of simple irregular problem and the results are presented in graphs.

# ABSTRAK

Dalam matematik, kaedah unsur terhingga adalah kaedah berangka bagi mencari penyelesaian anggaran untuk masalah nilai sempadan untuk persamaan pembezaan. Istilah 'Unsur terhingga' berpunca daripada prosedur di mana struktur yang dibahagikan kepada unsur-unsur bersaiz kecil tetapi terhingga. Kaedah unsur terhingga amat berguna untuk masalah berkaitan dengan geometri yang rumit, beban, dan sifat bahan di mana penyelesaian analisis tidak boleh diperolehi. Dalam kajian ini, masalah geometri tidak sekata untuk dua dimensi taburan haba dipilih sebagai satu contoh masalah industri yang perlu diselesaikan menggunakan kaedah unsur terhingga dan kaedah perbezaan terhingga. Pengaturcaraan Matlab digunakan sebagai medium pengiraan bagi kedua-dua kaedah tersebut. Disebabkan hasil yang didapati daripada pelaksanaan masalah geometri tidak sekata menggunakan kaedah unsur terhingga dan kaedah perbezaan terhingga menumpu , ia juga membuktikan bahawa hasil yang diperolehi adalah betul. Oleh itu, boleh disimpulkan bahawa masalah geometri tidak sekata boleh diselesaikan menggunakan kaedah unsur terhingga dan kaedah perbezaan terhingga  Melalui kajian ini, kaedah unsur terhingga didapati dapat menghasilkan jawapan yang lebih stabil dan konsisten jika dibandingkan dengan kaedah perbezaan terhingga dan semua hasil yang didapati dipersembahkan di dalam graf.

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| | | |
|---|---|---|
| $\delta$ | - | delta |
| $\psi^T$ | - | global virtual temperature vector |
| $T^e$ | - | displacement vector |
| $K^e$ | - | the stiffness matrix |
| $F^e$ | - | load vector |
| $Q$ | - | heat generation |
| $E_i$ | - | truncation error |
| $\pi$ | - | pi |
| $n$ | - | normal flux |
| $T$ | - | temperature |
| $A_e$ | - | area of the element |
| $\det(J)$ | - | determinant of Matrix $J$ |
| $S_T$ | - | specified temperature |
| $S_q$ | - | specified heat flux |
| $S_c$ | - | convection |

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# CHAPTER 1

# INTRODUCTION

## 1.0 Background of Study

The finite element method (FEM) is a computational or numerical technique which gives approximate solutions of boundary value problem arising normally in physics and engineering (Pepper and Heinrich, 1992). Chao and Chow (2002) stated that the fundamental idea of FEM is to discretise the domain into several subdomains, or finite elements. These elements can be irregular and possess different properties so that they form a basis to discretise complex structures, or structures with mixed material properties.

Further, they can accurately model the domain boundary regardless of its shape. Boundary value problems, sometimes called as field problems is a mathematical problem in which one or more dependent variables must satisfy specific conditions on the boundary of domain and satisfy a differential equation everywhere within a known domain independent variables. The examples of field variables are physical displacement, temperature, heat flux and fluid velocity depending on the type of physical problem being analysed. FEM cuts a structure into several elements (pieces of the structure) then reconnects elements at “nodes” as if nodes were pins or drops of glue that hold elements together. This process results in a set of simultaneous algebraic equations. According to Hutton (2004), the term finite element was first coined by Clough in 1960. In the early 1960s, and engineers used the method for approximate solutions of problems in stress analysis, fluid flow, heat transfer, and other areas.

Apart from FEM, Finite difference method (FDM) is a common numerical method for the solution of partial differential equations (PDEs) and ordinary differential equation (ODEs). FDM involves discretization of the spatial domain, the differential equation, and boundary conditions, and a subsequent solution of a large system of linear equations for the approximate solution values in the nodes of the numerical mesh.

In FDM one starts with the differential formulation and by a process of discretization transforms the problem into a system of interlinked simultaneous algebraic equations that then must be solved in order to determine an approximation to the desired solution. Discretization consists of first introducing a mesh of nodes by subdividing the solution domain into a finite number of sub domains and then approximating the derivatives in the boundary value problem by means of appropriate finite difference ratios which can be obtained from a truncated Taylor series expansion. As a result, system of interlinked simultaneous algebraic equations is obtained that then must be solved in order to determine an approximation to the desired solution (Reimer and Cheviakov, 2012).

For comparing the two methods, FEM actually models the differential equations and uses numerical integration to get the solution at discrete points while the FDM models the differential equation using finite difference formulas derive from truncated Taylor series expansion.

## 1.1 Problem Statement

A numerical method is a technique for obtaining approximate solutions of many types of engineering problems. The need for numerical methods arises from the fact that for most practical engineering problems their analytical solutions do not exist. While the governing equations and boundary conditions can usually be written for these problems, difficulties introduced by either irregular geometry or other discontinuities are difficult to be solved analytically. FEM and FDM are the examples of numerical methods that can be used to solve this kind of problems. Though FDM is easier to compute as well as to code, some problems that involved complex geometry is either difficult or completely cannot be solved by FDM. Generally FDM is a simple method to use for common problems defined on regular geometries, such as an interval in one dimension (1D), a rectangular domain in two dimensions (2D), and a cubic in three dimensions (3D). Meanwhile FEM is a complex method but it can be used to solve complex geometry problems. That is why FEM is essential for industrial problem computational purposes. Hence for this research, our aim is to solve 2D simple irregular geometry heat distribution problems using FEM and FDM and compare the result for both methods.

## 1.2 Objectives of Study

The objectives of this study are:

(I) To implement FEM and FDM for heat distribution problems.

(II) To develop the coding for 2D regular geometry heat distribution problem using FEM and 2D simple irregular geometry heat distribution problem using FEM and FDM in Matlab programming.

(III) To compare the results of FEM and FDM on 2D simple irregular geometry heat distribution problems.

## 1.3 Scope of Study

This study will emphasize on the fundamental theory of FEM and FDM and its application towards the solution for heat distribution problems especially for irregular geometry case and the implementation through Matlab. The Matlab coding is developed based on Laplace equation for irregular geometries using FDM whereas Galerkin method is used for solution related to FEM. The solution for FDM and FEM approaches ultimately end up with having system of linear equations. System of linear equation can be solved using many methods such as Gauss elimination method, LU decomposition method, QR decomposition method and Jacobi method. However, in this research we used Jacobi method since the method is an iterative method that first generates inexact results and subsequently refines its results at each iteration, with the residuals converging at an exponential rate giving values that are correct to specified accuracy (Fraser, 2008).

## 1.4 Significance of Study

This research benefits the student which gives them a better understanding about the application of FDM and FEM in preparing them to solve the real world of engineering problems. Other than that, this study will help engineers to solve many industrial practical problems particularly involved complicated domains. For instance, in a frontal crash simulation it is possible to increase prediction accuracy in important areas like the front of the car and reduce it in its rear (thus reducing cost of the simulation). This application can be extended to soil mechanics, heat transfer, fluid flow, magnetic field calculations, and other areas. The introduction of the digital computer has made possible the solution of the large-order systems of equations.

## 1.5 Organization of Research

There are five chapters in this research. In Chapter 1, we will discuss about the introduction of this research, problem statement, objective, scope and significance of the research. Next, we will present the literature review regarding FDM and FEM in Chapter 2. After that, we will introduce the mathematical formulation of the problem and explain theoretically the steps for the solution of the problems using FDM and FEM in Chapter 3. Then, in chapter 4 we will present the numerical results for FDM and FEM obtained from Matlab and discuss the results. Finally, in Chapter 5, we conclude our research and present some suggestions for future research.

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