NUMERICAL METHOD FOR SOLVING A NONLINEAR INVERSE DIFFUSION EQUATION

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Specially dedicated to my beloved mother and father,

Loumi Binti Giti and Jominis Bin Sinodian,

to my siblings

and

those people who have guided and inspired me throughout my journey of education

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ABSTRACT

Numerical Method is a way in solving a model of a problem mathematically and predicts the behaviour of the problem. That is why this method is very important for both natural and manmade process. Some basic theory of the Numerical Method has been applied in our daily life such as chemistry, physics, engineering, biology and others. The purpose of this project is to develop a numerical method for solving a one-dimensional inverse problem. In order to solve the equation problem, some assumptions such as the existence and uniqueness of the inverse problem, are taken into consideration where auxiliary problem and Schauder Fixed-point theorem were take place in order to prove it. Furthermore, a Numerical Algorithm such as Fully implicit Finite-different method and least square minimization method for solving a nonlinear inverse problem is proposed. At first, Taylor's series Expansion is employed to linearize the nonlinear terms and then the finite-different method is used to discretize the problem. The present approach is to rearrange the system of linear differential equation into matrix form and then estimate the unknown diffusion coefficient via Least-square minimization method. Computer programming namely Maple 13 will be used as an additional method to improve the accuracy between the exact solution of the problem and the result from comparing the numerical method with a exact solution. Lastly, the graphing of the curve based from the result obtained will be done by using Maple 13. Throughout the project, we can conclude that all three objectives have been achieved.

ASTRAK

Kaedah Berangka merupakan satu cara dalam menyelesaikan masalah model matematik dan meramalkan tingkah laku masalah. Oleh Sebab itu, kaedah ini adalah sangat penting bagi proses semula jadi dan buatan manusia. Beberapa teori asas Kaedah berangka telah digunakan dalam kehidupan harian kita seperti kimia, fizik , kejuruteraan , biologi dan lain-lain . Tujuan projek ini adalah untuk membangunkan satu kaedah berangka bagi menyelesaikan masalah songsang satu dimensi. Dalam usaha untuk menyelesaikan masalah persamaan, beberapa anggapan seperti kewujudan dan keunikan masalah songsangan, diambil kira dengan masalah tambahan dan teorem Schauder titik-tetap diperlukan untuk membuktikannya. Seterusnya dicadangkan, kaedah beza Terhingga tak tersirat digunakan untuk menyelesaikan masalah tak linear songsang. Pada mulanya , kembangan siri Taylor digunakan untuk melinearkan terma tak linear dan kemudian kaedah beza terhingga digunakan untuk mendiskritkan masalah. Pendekatan ini adalah untuk menyusun semula sistem persamaan pembezaan linear ke dalam bentuk matriks dan kemudian menganggarkan pekali resapan yang tidak diketahui melalui kaedah peminimuman kuasadua terkecil . Pengaturcaraan komputer Maple 13 digunakan sebagai kaedah tambahan untuk meningkatkan ketepatan antara penyelesaian tepat masalah ini dan keputusan yang membandingkan kaedah berangka dengan penyelesaian yang tepat. Akhir sekali, graf keluk berdasarkan dari keputusan yang diperolehi akan dilakukan dengan menggunakan Maple 13. Sepanjang projek ini, kita boleh membuat kesimpulan yang ketiga-tiga objektif telah dicapai.

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LIST OF ABBREVIATIONS

ODE	-	Ordinary Differential Equation
PDE	-	Partial Differential Equation
BVP	-	Boundary Value Problem
IVP	-	Initial Value Problem
RHS	-	Right Hand Side
LU	-	Lower Upper Decomposition algorithm

LIST OF SYMBOLS

T(t)	-	Temperature at time
$\frac{du}{dt}$	-	First derivative of u with respect of time
$\frac{du}{dx}$	-	First derivative of u respect of x
α	-	alpha, Unknown Diffusion Coefficient

CHAPTER 1

BACKGROUND OF STUDY

1.1 Introduction

The inverse problem and inverse nonlinear inverse parabolic problem have been previously treated by many authors who considered certain special case of this types of problem.[1]. In this project shall introduce the mathematical model of one-dimensional inverse nonlinear diffusion problem with the initial condition and boundary condition. In [2], the researcher mentioned about the mathematical model for nonlinear inverse diffusion problem which is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\alpha\big(u(x,t)\big)\frac{\partial u}{\partial x}\right), \qquad 0 < x < 1, 0 < t < T, \qquad \textbf{(1.1)}$$

$$u(x,0) = \varphi(x), \qquad 0 < x < 1, \qquad \textbf{(1.2)}$$

$$\alpha\big(u(0,t)\big)\frac{\partial u(0,t)}{\partial x} = g_0(t), \quad 0 < t < T, \qquad \textbf{(1.3)}$$

$$a\big(u(1,t)\big)\frac{\partial u(1,t)}{\partial x} = g_1(t), \qquad 0 < t < T, \qquad \textbf{(1.4)}$$

$$u(0,t) = f(t), \qquad 0 < t < T, \qquad \textbf{(1.5)}$$

where $\alpha(u(x,t))$ is known as coefficient of the inverse nonlinear diffusion equation and always be positive function and also the term which remain to be determined., T is a given positive constant, $\varphi(x), g_0(t), g_1(t), f(t)$ are piecewise-continuous known function. In this problem, the researcher considered a horizontally placed cylinder of unit length which is filled with a homogenous porous material and parallel to the axis of the cylinder there is a liquid flow which is partially saturates to the porous medium [2]. As the researchers study the equation, they considered the problem (1.1)-(1.4) as a direct problem, where the coefficient $\alpha(u(x,t))$ will be determined from the overspecified data (1.5). Equation (1.1) came from the combination of the Darcy's Law and Continuity equation [3] where conditions (1.3) and (1.4) represent the flux of the moisture at the position $x = 0$ and $x = 1$ respectively and conditions (1.2) and (1.5) represent the moisture content at $t = 0$ and in the position $x = 0$.

What is mathematical method? Mathematical method is a process in solving, describing and proving an equation or a problem in term of mathematic through various methods that are applicable in mathematic field. There are many methods that can be applied in solving a problem whether analytical method or numerical method. Thus, in case to solve one problem it is very importance to use more than one method to make sure there is no error during our calculation process according to M.Rahman[4]. analytical solution only exist for some of the problem but in others, the difficulty in obtaining analytic solution is due to the governing of differential equation and some from the complexities of geometrical configuration of physical problem.

What is computer programming? Computer programming is a process of developing and implementing various sets of instructions to enable a computer to do a certain task. These instructions are considered computer programs and help the computer to operate smoothly.

1.2 Problem Statement

Nonlinear inverse diffusion equation has been used wisely not only in the science field but also in technology area as well. Hence, analysis implementation and testing of inverse algorithms are also one of the great scientific and technological interests but in general finding and research about inverse problem, they had proven that the inverse problem has more than one solution and it is called ill-posed. Where, the solution of the inverse problem does not satisfy the general requirement if they were applying a small change in input data. The mathematical approach of the equation could help us understand more detail about the equation itself and make it easier to be used in future needs.

This research will explore on how to estimate the unknown coefficient of the nonlinear inverse diffusion equation with given additional information and characteristic of each of the variables and function. The first problem that we want to conduct is how to estimate the unknown diffusion coefficient using numerical approach, to find the good example in real life problem especially in engineering problem that related to the research and then finally we will try to describe the behaviour that represent the equation via computer programming.

1.3 Objective Of The Study

The objectives of this study:-

1) To approximate the suitable value of unknown diffusion coefficient from the inverse nonlinear diffusion equation via numerical approach such as Taylor's series expansion, Finite-Difference method and Least-Square Minimization.
2) To relate the current method to the Groundwater Hydrology problem(Numerical experiment)
3) To describe the result of numerical experiment via computer programming such as MAPLE 13.

1.4 Scope Of The Study

This project is focus on how to solve or in specifically to estimate the unknown coefficient of the linear inverse diffusion equation since the solution of the inverse nonlinear diffusion equation requires us to determining an unknown diffusion coefficient. This research also concentrates on finding way to solve the equation that represent in the engineering problems. Process of solving the equation needs to refer more than one aspect so that the result of the equation will be more accurate when we apply a change in the input data. Process in solving of the equation must firstly focus on showing the uniqueness and existence [2] with assist from some theory, theorem and also lemma from previous research where the function along with the initial and boundary condition must be continuous and satisfy some other condition in order to prove them unique and exist. For example, based from Kantorovich[5]

‘if U is a convex subset of a topological vector space V and R is a continuous mapping of U into itself so that $R(U)$ is contained in a compact subset of U, then R has a fixed point.”

Before we proceed to the next process, we must need to consider the function and the variable in order to avoid any errors and make things clear during the process of solving the equation and the process will begin by first assuming the unknown coefficient diffusion. There are no prior information is available on the functional form of the unknown diffusion coefficient in this project [2]. Thus, we will classify the unknown coefficient as the function estimation in the inverse calculation. In other words, the unknown coefficient in our project acts as the stabilizing of the inverse problem. For the unknown diffusion coefficient $\alpha(u(x,t))$, we must therefore provide the additional information to provide a unique solution for the nonlinear inverse diffusion equation. So, the additional information is usually given by adding small random error to the exact value from the solution to the direct problem.[2].

The application of the nonlinear inverse diffusion equation will be solve using the following step. Firstly, we will transform the nonlinear inverse diffusion equation into linear equation or we called it linearization method by using Taylor’s series expansion. Linearization sometimes refer to finding the linear approximation to the function at the given point, it is a method for assessing the local stability of equilibrium of the nonlinear differential equation.

After linearization process, the result equation will be discretize by the certain method in numerical field namely fully implicit finite different method. The present approach is to rearrange the matrix forms of the differential governing equations and finally we will use least square minimizing method to estimate the unknown diffusion coefficient and in this current step the additional information will be used respectively. The result then will be tested in to real life problem such as in engineering problem and also will be describe by using computer programming such as Maple13.

1.5 Significant Of The Study

Inverse problem had been used wisely in recent years not only one of the way to solve problem in daily activities, for engineering application such that designing machine, and any designing that related to the wave problem. In the past years, the demands of application have been becoming much stronger, whereas, until recently most engineers could get by with linear differential equation as well as partial differential equation as key to model the problem in their work problem, it is shown that the mathematical knowledge has been used as a solution discovered by researcher which then will be used to approximate the solution of the phenomena.

1.6 Thesis Organization

This thesis is organized into five chapters. First chapter contains the study of framework. Background of the problem, statement of the problem, objective, scope and significance of the study are discussed in this chapter.

In chapter two, present some details about diffusion equation, differential equation whether on linear and nonlinear equation, also literature review of the most application of first order nonlinear inverse diffusion equation will also be discussed in this chapter.

In chapter three, the methodology will be discussed. This chapter introduces the sequence of the process in solving the nonlinear inverse diffusion equation. The theory, theorem and also lemma will be stated and explained in detail to extract the information in order to solve the equation. Lastly, the first objective of the project will be achieved in this chapter.

In chapter four, result and discussion will be highlighted. The result from the previous chapter will be tested in real-life problem. Thus, in order to achieve our second objective we will choose one of the problems in area of engineering and solve it via suggested method. In the end, the numerical result from real life problem will be described via computer programming and the software that we choose to solve the problem will be MAPLE13.

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