

**MESHFREE FORMULATION OF GEOMETRICAL NONLINEAR
COMPOSITE BEAM WITH PARTIAL INTERACTION**

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MESHFREE FORMULATION OF GEOMETRIC COMPOSITE BEAM WITH
PARTIAL INTERACTION

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In the name of Allah, the Most Beneficent, the Most Merciful.

This thesis is dedicated to my mother, Samsiah Binti Ramli and my father, Ardianshah Talib.

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ABSTRACT

This study concerns with the formulation of Meshfree (MFree) for nonlinear geometric of composite beam with partial interaction. The Principle of Virtual Work was used to derive the differential equation of composite beam. Finite Element Method (FEM) and MFree method: Point Interpolation Method (PIM) was used to solve the differential equation. The derived formulation was validated with previous research work for linear problem and nonlinear problem. The nonlinear geometrical are taken into account to study the performance of Mfree handling the nonlinear problem and the performances are compared with FEM. The algorithms of the solution procedure for both methods were written in MATLAB. Parametric studies were conducted to study the performances in term of convergence rate and computer resources between FEM and PIM. The parameters considered in this study were the size of support domain, α_s number of nodes, number of Gauss cell and number of Gauss point. The result of the parametric study showed that five bending nodes and nine axial nodes with α_s equal to four give the appropriate result considering both accuracy and stability. The recommended value for Gauss cell was three and the number of Gauss point was five. Two parameters observed to study the use of computer resources were the computational speed and memory used to solved the problem. From the study, MFree has been found to have a potential as FEM yet another option of numerical method in solving engineering problem in general and composite beam problem in particular. However, further studies are required in improving the efficiency of the method specifically in regards to the high consumption of computer resources.

ABSTRAK

Kajian ini adalah mengenai formulasi Jaring Bebas (MFree) bagi analisis geometri bukan linear rasuk rencam dengan interaksi separa. Prinsip Kerja Maya telah digunakan bagi menerbitkan persamaan pembezaan rasuk rencam. Kaedah Unsur Terhingga (FEM) dan kaedah MFree: Kaedah interpolasi titik (PIM) telah digunakan untuk menyelesaikan persamaan pembezaan. Persamaan yang diterbitkan telah disahkan dengan dengan hasil penyelidikan yang lepas bagi masalah linear dan masalah bukan linear. Kesan analisis geometri bukan linear diambil kira didalam kajian ini adalah bagi mengkaji prestasi Mfree dalam menyelesaikan masalah bukan linear dan membandingkan keputusannya dengan FEM. Algoritma prosedur penyelesaian bagi kedua-dua kaedah telah ditulis menggunakan perisian MATLAB. Kajian parametrik dijalankan untuk mengkaji prestasi dalam kadar penumpuan dan sumber komputer antara FEM dan PIM. Parameter yang dipertimbangkan dalam kajian ini adalah saiz domain sokongan, α_s bilangan nod, bilangan sel Gauss dan bilangan titik Gauss. Hasil kajian parametrik menunjukkan bahawa lima nod rasuk dan sembilan nod paksi dengan α_s bersamaan dengan empat memberikan hasil yang sesuai berdasarkan ciri ketepatan dan kestabilan. Nilai yang disyorkan bagi sel Gauss adalah tiga dan bilangan titik Gauss adalah lima. Dua parameter diperhatikan untuk mengkaji penggunaan sumber komputer adalah kelajuan pengiraan dan memori yang digunakan bagi menyelesaikan masalah. Dari segi kelajuan, FEM lebih cepat berbanding Mfree manakala untuk penggunaan memori, Mfree adalah kurang daripada FEM. Daripada kajian, didapati Mfree mempunyai potensi sebagaimana FEM dan merupakan pilihan lain untuk kaedah berangka bagi menyelesaikan masalah kejuruteraan umumnya dan dalam masalah rasuk rencam khususnya.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS:

E_s	-	concrete modulus of elasticity
E_b	-	steel modulus of elasticity
I_s	-	concrete moment of inertia
I_b	-	steel moment of inertia
A_s	-	concrete cross-sectional area
A_b	-	steel cross-sectional area
F	-	Forces
s	-	shear connectors spacing along the beam
k	-	shear connector modulus
C_s	-	the distance from the contact surface to the centroidal axis of concrete section
C_b	-	the distance from the contact surface to the centroidal axis of steel section
M_s	-	concrete section resisting moments
M_b	-	steel section resisting moments
z	-	the distance between the centroidal axis of the concrete slab and steel beam
u	-	axial displacement between the concrete and steel sections
s	-	slip
h	-	distance between the centroid of concrete and steel
y_1	-	distance from the centroid of concrete to interlayer surface
y_2	-	distance from the centroid of steel to interlayer surface
ε	-	strain

κ	-	curvature
W	-	work done
N	-	normal force
M	-	bending moment
σ	-	stress
w	-	displacement
∂	-	differential operator
δ	-	variation
g	-	load vector
k	-	tangent stiffness
C	-	constitutive
K	-	connection tangent stiffness
d_s	-	size of support domain
α_s	-	dimensionless size of support domain
d_c	-	nodal spacing
u_c	-	concrete axial displacement
u_s	-	steel axial displacement
Ω	-	Global Problem Domain
Φ	-	Matrix of Shape Functions
G_b	-	Bending Moment Matrix of Composite Beam
G_c	-	Axial Moment Matrix of Concrete
G_s	-	Axial Moment Matrix of Steel
U	-	Nodal Displacements Vector
U_b	-	Deflection And Slope Displacements Vector
U_c	-	Axial Displacement Vector of Concrete
U_s	-	Axial Displacement Vector of Steel

ABBREVIATION :

EFG	-	Element Free Galerkin
FEM	-	Finite element method
MFree	-	Meshfree
PCFC	-	Precast Cold-Formed Composite
PIM	-	Point Interpolation Method
WLS	-	Weighted Least Square

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

Composite structures especially composite beams are widely used in today's construction. Compositing beams are composed of two or more materials joint together to act as a single unit. Steel-concrete composite beams are one of the common examples of composite beam which often used for long-span beams in buildings and bridges. Concrete is strong in compression but weak in tension and steel, on the other hand better in tension than compression. The steel and concrete elements are tied together using shear connectors. Headed stud shear connectors are most commonly used as the shear connectors. The stud usually welded to the top flange of a steel beam to resist longitudinal slip and vertical separation between the concrete slab and steel beam. Moreover, by providing suitable shear connectors between the steel and concrete interface will assist in increasing the load carrying capacity of the composite beam due to increment of the shear resistance of interface Newmark *et al.* (1951).

There are many types of shear connectors and most generally divided into rigid and flexible shear connectors. For rigid shear connectors, the composite beam exhibits full interaction whilst for flexible shear connectors, the composite beams exhibit partial

interaction. If the shear connectors are not rigid, small longitudinal as well as interlayer slip occurs between the elements.

Commonly, most of the problems of composite beam with interlayer slip are assumed as linear. For linear analysis, when the force applied, the displacement is assumed linear but in reality the problems exhibit nonlinear behavior. As load increase, there are progressively changes of the stiffness of a structure, as a result of material changes, and/or geometric and contact effects. There are three common sources of nonlinearity; geometric nonlinearity, material nonlinearity and boundary condition nonlinearity. For geometric nonlinearity, the stiffness of structures is dependent on the displacement. The geometrical effects may be unexpected, thus the analysis may fail to give the real structural behavior if the effect is not taken into consideration. Material nonlinearity refers to nonlinear stress-strain response and often resulted by the gradual weakening of the structural behavior when the load is increased. For boundary condition nonlinearity comes from the effect of geometrical nonlinearity. For this study, only geometric nonlinearity is included and to be discussed, while the other two nonlinear are excluded.

1.2 PROBLEM STATEMENT

Very frequently, the partial differential equations for engineering problems are so complicated that their solution in close form is either impossible or impracticable. Hence one has to resort seeking numerical solution. Finite Element Method (FEM) is one of the most general and powerful technique for the numerical solutions and widely used in engineering analysis. FEM have been used by other researchers like Porco *et. al.* (1994), Salari and Spacone (2001) and Silva and Sousa (2010) to solve composite beam problems. However, the accuracy of FEM is decrease when dealing with distorted

element or crack propagation problem since the FEM procedure is relies on predefined mesh or element. Thus the mesh refinement is needed for the problem and such procedures can be complex and time consuming for computer-based analysis. Since the Meshfree (MFree) is not relying on any predefine mesh, it seems to have the potential to overcome the drawback of FEM. From the literature review, the applications of MFree in structural engineering problems are not compressive. There are several interpolation techniques of MFree such as Moving Least Square (MLS), Point Interpolation Methods (PIM), Radial Point Interpolation Methods (RPIM) and Patition of Unity (PU) methods. Hence, the PIM will be used for this study in solving geometric nonlinearity of the composite beam problem.

1.3 PURPOSE AND OBJECTIVES OF THE STUDY

The purpose of this study is to use the PIM in solving geometric nonlinearity problem of the composite beam. The objectives of the study are listed at below:

1. To formulate geometric nonlinearity problem of partial interaction of composite beam using PIM.
2. To verify obtained results with exact or other numerical solution.
3. To obtain the optimum parameters used in PIM in solving the problem.

1.4 SCOPE OF THE STUDY

The PIM is employed in solving the geometric nonlinearity problem of composite beam with partial interaction.

The assumptions and limitations of this study are listed as below:

1. Materials are assumed linear elastic.
2. Large transverse displacements, small strains and small to moderate rotations.
3. It is assumed equal curvatures for both steel and concrete.
4. The problem is analyzed as one-dimensional element.
5. Concrete is assumed as uncracked.

1.5 SIGNIFICANCE OF THE STUDY

MFree have been reported to have better accuracy than FEM due to the use of higher order polynomials for the trial functions. MFree also has an advantage over the FEM, because of its capability of handling deformation resulted from geometrical nonlinearity. The availability of MFree in solving geometric nonlinearity of composite beam with slip problem will provide great flexibility to numerical analysis of problems on composite structures. In the next chapter, several advantageous of using this method to solve mechanics related problem will be detailed. Although there are many assumptions made for the simplicity of this study, it is believed that this study will provide a good review for future works.

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