IRREDUCIBLE REPRESENTATION OF FINITE METACYCLIC GROUP OF NILPOTENCY CLASS TWO OF ORDER 16

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ABSTRACT

Representation theory is a study of real realizations of the axiomatic systems of abstract algebra. It originated in the study of permutation groups, and algebras of matrices. Representation theory has important applications in physics and chemistry. This research focuses on finite metacyclic groups. The classification of finite metacyclic groups is divided into three types which are denoted as Type I, Type II and Type III. For any group, the number of possible representative sets of matrices is infinite, but they can all be reduced to a single fundamental set, called the irreducible representations of the group. Irreducible representation is actually the nucleus of a character table and is of great importance in chemistry. In this research, the irreducible representation of finite metacyclic groups of class two of order 16 are found using two methods, namely the Great Orthogonality Theorem Method and Burnside Method.

ABSTRAK

Teori perwakilan merupakan satu kajian mengenai kesedaran nyata bagi sistem aksiom dalam aljabar abstrak. Ia berasal daripada kajian dalam bidang kumpulan pilihatur dan aljabar dalam matriks. Teori perwakilan mempunyai aplikasi penting dalam bidang fizik dan kimia. Penyelidikan ini tertumpu kepada kumpulan metakitaran terhingga. Klasifikasi bagi kumpulan metakitaran terhingga dibahagikan kepada tiga jenis yang ditandai sebagai Jenis I, Jenis II dan Jenis III. Untuk sebarang kumpulan, bilangan yang mungkin bagi set perwakilan matriks adalah tak terhingga, tetapi ia boleh terturun kepada satu set asas yang dipanggil perwakilan tak terturunkan bagi kumpulan. Perwakilan tak terturunkan sebenarnya adalah nukleus bagi jadual aksara dan ia sangat penting dalam bidang kimia. Dalam penyelidikan ini, perwakilan tak terturunkan bagi kumpulan metakitaran terhingga kelas dua berperingkat 16 telah ditemui menggunakan dua kaedah yang dikenali sebagai Kaedah Teori Agung Ortogon dan Kaedah Burnside.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

G	A group G
$Z(G)$	Center of the group G
$\chi_i(R)$	Character of the representation of R in the $i-th$ irreducible representation
χ_i^j	Character of elements in class C_i in the irreducible representation labelled by j
d_k	Character of the irreducible representation
$C_{ij,s}$	Class multiplication coefficients
l_i	Dimension of the $i-th$ representation
$\in$	Element of
G/H	Factor group
$G \cong H$	G is isomorphic to H
$>$	Greater than
$\geq$	Greater than or equal
$\langle x \rangle$	Group generated by x
$H \leq G$	H is a subgroup of G
1	Identity element
Γ_i	$i-th$ irreducible representation
δ_{ij}	Kronecker's Delta
aH	Left coset
$<$	Less than
$\leq$	Less than or equal
$[g,h]$	The commutator of g and h
${}^g h$	The conjugate of h by g
$K(G)$	Number of conjugacy class
$\|G\|$	Order of the group G
h	Order of a class

CHAPTER 1

INTRODUCTION

1.1 Background of the Research

According to Curtis and Reiner [1], representation theory has been developed in 1896 in the work of the German mathematician named Frobenius at the end of the nineteenth century. In general, representation theory is a study of real realizations of the axiomatic systems of abstract algebra. It originated from the study of permutation groups, and algebras of matrices. It is a beautiful mathematical topic which has many useful applications. Group representation has also a big role in the theory of finite abstract groups. The first systematic book on representation theory has emerged in 1911 and includes many results which were prove using group characters on abstract groups [2]. Perhaps the most important of these is the Burnside's theorem that says a finite group whose order has at most two distinct prime divisors must be solvable. Recently, a purely group-theoretic proof of Burnside's theorem has been obtained[1].

According to Curtis and Reiner [1], another stage in the development of representation theory started in 1929, when Noether resulted in the assimilation of the theory into the study of modules over rings and algebras. Representation theory was an evolution of modular representation of finite groups. Like the original work of Frobenius, Brauer's theory has many significant applications to the theory of finite groups.

During the past century, there has been increased interest in integral representations of groups and rings, motivated to some extent by questions arising from homological algebra. This theory of integral representations has been a fruitful source of problems and conjectures both in homological algebra and in the arithmetic of non-commutative rings.

However, representation theory is much more than just a mean to study the structure of finite groups. It is also a basic tool with applications in many areas of mathematics and statistics, pure and applied together. For example, sound compression is very much based on the fast Fourier transformation form of finite Abelian groups. Fourier analysis on finite groups has also an important role in probability and statistics, especially in the study of random walks on groups. Also, there are some applications of representation theory in graph theory, and in particular to the construction of expander graphs. In the theory of groups itself, linear representations are an irreplaceable source of examples and a tool for investigating groups.

A linear representation of the group G over the field K is a homomorphism of G into the group $GL(V)$ of all invertible linear transformations (linear operators) of a vector space V over K. The vector space V is called the representation space, and its dimension is called the dimension or the degree of the representation.

This research focuses on metacyclic groups. The study of metacyclic groups has been done in [3-5]. In this research, the irreducible representation of finite metacyclic groups is determined. These include the structure of irreducible representations, characters, and irreducible components for finite metacyclic groups.

1.2 Statement of the Problem

The study of irreducible representation has been done for many groups including symmetric group by Murnaghan [6], groups of order 8 by Sarmin and Fong [7] and finite classical groups by Lusztig [8]. However, it has not been done for metacyclic groups. Therefore our statement of problems is stated in the following:

How does the irreducible representation of finite metacyclic groups look like? What is the structure of group representations, specific irreducible representations and components for finite metacyclic groups?

1.3 Objectives of the Research

The main objectives of this research are:

i. to study the representation theory in general and the irreducible representations of groups,
ii. to study the finite metacyclic groups and their characteristics,
iii. to find the irreducible representations of finite metacyclic group of class two of order 16 using Great Orthogonality Theorem method,
iv. to find the irreducible representations of finite metacyclic group of class two of order 16 using Burnside method.

1.4 Scope of the Research

Certain classes of metacyclic groups have been given much emphasis by many authors. In 2005, Beuerle [9] classified the non-abelian metacyclic *p*-groups of class at least 3 where *p* is prime. The irreducible representation of finite metacyclic groups has not been done yet. Thus, in this research, the irreducible representation of finite metacyclic groups of class two of order 16 will be given. This research only focuses on obtaining the irreducible representation and the structure of the irreducible representation of some finite metacyclic groups.

1.5 Significance of the Research

The study of representation of finite groups is motivated by a number of applications in Natural Sciences. It is a beautiful mathematical topic which has many applications, ranging from number theory and combinatorics to geometry, probability theory, quantum mechanics and quantum field theory.

There are also many important applications of representation theory in physics, chemistry, and crystallography. For example, in physics, how the symmetry group of a physical system affects the solutions of equations describing that system can be described. In addition, representation theory of the symmetric group is a particular case of the representation theory of finite groups, for which a concrete and detailed theory can be obtained. This has a large area of potential applications, from symmetric function theory to problems of quantum mechanics of a number of identical particles. Besides, the application can also be found in quantum computing [10].

Furthermore, the representation of some types of metacyclic groups (namely dihedral groups) has many useful applications. One of them have been given by Lenz [11] who used representations of dihedral groups in the design of early vision filters. So, the results of this study will become a very useful tool to be used in many areas of application.

1.6 Conclusion

In this chapter, a brief history about representation theory was stated in the background of the study, where we provided some early and recent publications related to representation theory and irreducible representation. In addition, our objectives and statement of research problem are stated in this chapter. This research covered some topics which are stated in the scope of the research. The motivations and importance of this research are mentioned in the significance of the research. In the next chapter, some literature review related to this research are given.

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