# FINITE DIFFERENCE METHOD AND FINITE VOLUME METHOD FOR SOLVING NAVIER-STOKES EQUATION IN SEA WATER MOVEMENT

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A dissertation submitted in partial fulfillment
of the requirement of the award of the degree of
Master of Science (Mathematics)

Department of Mathematics
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Universiti Teknologi Malaysia

June, 2013

## ACKNOWLEDGEMENT

I would like to say Alhamdulillah and thank all people who have helped and inspired me during my study.

First and foremost, I would like to record my gratitude to my honorable supervisor, Assoc Prof Norma Alias for her supervision, advice, and guidance from the very early stage of this research as well as giving me extraordinary experiences throughout the thesis. The most needed, she provided me persistent encouragement and support in various ways. I am truly grateful for having such a wonderful supervisor.

I would like to extent my appreciation my family who always support me in my journey of education. They have given me so much comfort, care and love, either financially or spiritually, of which word could not express and will forever be remembered in my heart.

Lastly, I would like to thank a lot to my course mates and other friends especially Nor kamariah binti Kasmin@Bajuri and Anis Suhaida Ali who have provided their support and assistance to enable the completion of this research.

## ABSTRACT

This project proposed the Navier-Stokes equation of sea water modeling. The mathematical modeling consists of momentum, heat and salinity equations. The three parameters influence on sea water movement are momentum, temperature and salinity. The discretization of Navier-Stokes equation is based on Finite Difference Method (FDM) and Finite Volume Method (FVM). The numerical methods for solving FDM are JB and GS schemes. The results are computed by using Comsol Multiphysics Software (4.3a) and MATLAB 2011a. The numerical analyses are used to evaluate the comparisons of the FDM and FVM. The computational platform is Intel@ Core TM 2 Duo processor T7500 memory. The aspects of evaluation are number of iteration, time execution, maximum error and root square error and computational complexity. The results show that FDM is alternative method in visualizing the movement of the sea water.

# ABSTRAK

Projek ini mencadangkan persamaan Navier-Stokes model air laut. Pemodelan matematik itu terdiri daripada momentum, haba dan persamaan kemasinan. Ketiga-tiga pengaruh parameter pada pergerakan air laut momentum, suhu dan kemasinan. Bentuk pendiskretan persamaan Navier-Stokes adalah berdasarkan Kaedah Perbezaan Terhingga (FDM) dan Jumlah Kaedah Terhingga (FVM). Kaedah berangka bagi menyelesaikan FDM adalah JB dan GS skim. Keputusan dikira dengan menggunakan Comsol Multiphysics Software (4.3a) dan MATLAB 2011a. Analisis berangka digunakan untuk menilai perbandingan antara FDM dan FVM. Platform pengiraan adalah Intel @ Teras TM 2 Duo T7500 ingatan. Aspek penilaian yang diambil kira adalah bilangan lelaran, masa pelaksanaan, kesilapan maksimum dan akar ralat persegi dan kerumitan pengiraan. Keputusan menunjukkan bahawa FDM adalah kaedah alternatif dalam menggambarkan pergerakan air laut.

# TABLE OF CONTENTS

# LIST OF THE TABLE

# LIST OF FIGURES

# LIST OF SYMBOLS

| | |
|---|---|
| $u$ | Inertial Velocity |
| $\upsilon$ | Density |
| $p$ | Pressure |
| $\tilde{\sigma}$ | Deviatory (Viscous) Part Of The Stokes Tensor |
| $b$ | Body Force |
| $T$ | Temperature |
| $c_p$ | Heat At Constant Pressure |
| $\bar{q}$ | Heat Flux Vector |
| $\dot{q}$ | Heat Generation |
| $S$ | Salinity |
| $\bar{p}$ | Saline Diffusivity |
| $\dot{p}$ | Saline Generation |

# LIST OF APPENDIXES

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

Antarctica is the most southerly continent of the world (Scotese, C.R., *et al*. 1990). This project will investigate the sea water flow and the increasing of the sea level by Ishii, M. *et al* (2006) due to the increasing of the sea water in coastal Antarctica.

Sea water flow at the Coastal Antarctica region is continues and never stop. Sometimes, the flow also generated from the breaking waves, wind, temperature and salinity (Fuhrman, J.A., *et al.* 1980). The sea water is always moving. Same with the other thing, all the movement will influence by the gravity.

The mathematical modeling based on Navier-Stokes Equations is used to observe this phenomenon. Navier-Stokes Equations mostly used to describe the movement or motion of fluid substance (Girault, V., *et al*. 1986). These equations can be analyzed the momentum, temperature, salinity and pressure changes, gravity force, and other forces acting on the fluid flow. In addition, heat equation is investigated the temperature behavior of the sea water. This research will investigate the ocean flow using mathematical modeling called Navier-Stokes Equation that contains heat equation also salinity equation. These all equations are the partial differential equation. These equations can be applied in real phenomena in life science. Salinity is a concentration affected on sea water condition.

The data from National Institute of Water and atmospheric research Wellington University, New Zealand (NIWA) is used in this project. NIWA was formed in 1992 as a standalone company. It is a part of government initiatives to restructure the science sector in New Zealand. Many commercial and non-commercial researches across a broad range of disciplines in the environmental science have been conducted by it. The mission is to conduct the leading environmental science to enable the sustainable management of natural resources for New Zealand and the entire planet. Until 2008, about 753 staffs working in 15 sites in New Zealand and 1 in Perth, Australia. The Head office of NIWA is in Auckland. The data from NIWA was used in this project as validity to the approximate results obtained using Navier-Stokes Equation.

## 1.2 Background of the Problem

Mathematical modeling of the sea water movement components consist of several equations. The flow of the sea water is described by the Navier- Stokes Equation (Beddhu, 1994). The Navier-Stokes Equation is widely used in the fluid dynamics Ferziger, J.H *et al* (1996). Since in Navier- Stokes Equation can be presented the velocity changes then we can predict the velocity of the sea water for future climate change. In addition, heat equation important in investigating the heat transfer of the sea water flow. Heat transfer is the movement from the high temperature to the low temperature. Lastly, salinity equation usually is used to monitor the salinity as a concentration affected on sea water condition (Gordon, 2000). All the dependent variables for sea water movement visualization are velocity, temperature and salinity. Those parameters are used to investigate the sea water flow.

Finite Difference method (FDM) is a numerical strategy for discretizing the Navier- Stokes Equation. An approximate derivative is producing the approximation solution of the sea water flow. FDM in particular, are relatively straight forward and easy to implement on Navier- Stokes Equation. FDM involve approximations of derivatives at discrete computed nodes in a mesh of a structured domain. The types of the FDM used are centre and backward type. Using the FDM, the result in terms of graph can be simulating. FDM is applied directly in the Navier- Stokes Equation to get the discrete form.

Besides that, the Finite Volume Method (FVM) is widely used in the computational fluid dynamic. While in heat equation, the FVM is the best method to apply in fluid movement (Murthy, J.Y., *et al*, 1998).In 1999, finite volume method has been used for advection-diffusion equations in irregular geometries (Calhoun, 1999). The algorithm use the Cartesian grid in which some cells are cut by the embedded boundary. Then, FVM was used in heat equation model (Fang, 2009). Besides, FVM was used to discretize the operator, on the control volumes formed by intersecting the Cartesian grid cells with the domain, combined with a second-order accurate discretization of the fluxes of the heat equation (Schwartz, 2005).

This project wills shows how accurate FVM for solving the two dimensional problems. For the two dimensional problems, FVM is said to be more straightforward task (Zhang, 2012). For the 2 -dimensional geometry radiative heat transfer the method is stable and convergent. The results indicate that good accuracy is obtained on coarse computational grids. Besides that, the solution errors rapidly diminish as the grid is refined (Feldheim, 2007). Then, in fluid flow, the fluxes are calculated only on two-dimensional surfaces of the control volume instead of on three-dimensional space. The same reason applied in heat equation (Chao, 2002).

## 1.3 Problems Statement

The problem statement of this research is to investigate the accuracies of FDM and FVM for solving the sea water movement modeling by Kireev (2009). The performances of FDM and FVM methods are also under consideration. The aspects of the performance evaluations are iteration, time execution, maximum error and root square error, accuracy, and computational complexity.

## 1.4 Objectives

The objectives of this research are

1. To apply the Navier-Stokes Equation for investigating the sea water flow.
2. To discretize the Navier-Stokes Equation based on FDM using Jacobi method (JB) and Gauss Seidel (GS).
3. To discretize the Navier-Stokes Equation using FVM.
4. To analyze the numerical performances of numerical methods and its visualization using Comsol Multiphysics Software (4.3a)

## 1.5 Scope of Research

This research will focus on the Navier-Stokes Equation based on 3 parameters; momentum, temperature and salinity characteristics of the sea water region. The research scope is to obtain the discrete form of Navier-Stokes Equation using FDM and FVM. FDM is solved using JB and GS method. The aspects of the comparison evaluation are iteration, time execution, error and root square error, and computational complexity. Then, the comparison of FDM and FVM are applied to Navier-Stokes Equation.

Since, there is an actual solution from the NIWA, the comparison between the actual solution and some numerical methods will be analyzed. MATLAB 2011a are used to simulate and to capture the graph of momentum, temperature and salinity. The visualization of sea flow can be captured by Comsol Multiphysics Software (4.3a).

## 1.6 Hypothesis/Significant

The significant of this research is to capture the movement of the sea water using Navier-Stokes Equations including heat equation and salinity equation. MATLAB 2011a and Comsol Multiphysics Software (4.3a) are used to visual the sea water movement by changing the independent parameters.

The visualization is based on two methods; Finite Difference Method (FDM) and Finite Volume Method (FVM). Numerical performance measurements will use to analyze the comparison of two methods .The hypothesis is to prove FDM is better than FVM in two dimensional cases in visualizing the sea water movement.

FDM is simple and efficient in a structured mesh case compare to the FVM. Compared to FVM, it is easy to introduce higher order scheme of convection term.

## 1.7 Organization of Dissertation

Chapter 1 contains general introduction of the mathematical model and numerical methods of Navier- Stokes Equation. The background, objectives, statement and the significance of the research are also stated.

Chapter 2 defines the Navier- Stokes Equation. Numerical results are given for the Stokes and the Navier-Stokes Equations. Finally the method is applied to a double- diffusive convection problem concerning the stability of a fluid stratified by salinity and heated .Then followed by review some literature involves for the mathematical modeling that is under consideration. The method for discretization the mathematical modeling that is FDM and FVM also stated. Numerical analysis and computational platform system are also discussed. At the end of this chapter, the survey of references and the summary of literature review under consideration are organized in the table.

Chapter 3 begins with discretization of the Navier- Stokes Equation heat equation and salinity equation by using finite difference method (FDM). The entire steps taken from the beginning to the end have shown clearly. The sequential algorithm for each equation's step is showed.

Followed by Chapter 4, the FVM is implemented in the heat equation. The entire steps taken from the beginning of the equation to the end shown clearly.

Chapter 5 discusses the numerical results and comparison of FDM and FVM as described in chapter 3 and chapter 4. The results are compared to the exact solution of NIRW data based. A few tables and graphs are used to discuss the comparison of FDM and FVM. The performance comparison between GS and JB are also explained in chapter. Some performance results will be drawn to investigate the differences between two methods.

Finally, this report will end with Chapter 6 where the conclusion will be made and some recommendations for future research are provided. Then it is followed by references and appendices.

# REFERENCES

Abdullah,K.M., Abdullah, M.Z., Kamaruddin,S.,Ariff, Z.M., (2007), Study of flow visualization in stacked-Chip Scale Packages (S-CSP), *International Communications in Heat and Mass Transfer*, 34, 820-828.

Adcroft ,A. Jean-Michell Campin, Hill, C. and Marshall, J., (2004). Implementation of an Atmosphere–Ocean General Circulation Model on the Expanded Spherical Cube, *American Meteorological Society*. 132, 2845-2863

Barrett ,R., Berry, M., Chan, T. F., Demmel, J., Donato, J. M., Dongarra, J., Eijkhout ,V., Pozo, R., Romine, C., and Vorst, H. V.,(2000). *Templates for the Solution of Linear Systems:Building Blocks for Iterative Methods.* Second Edition.

Bedhu,M., Jiang ,M., Taylor K., and Whitfield, L. (1994). Towards Computations of Ocean Flows Using Navier-Stokes Equations. 144-153

Blessing, R.H., (1987), Data Reduction and Error Analysis for Accurate Single Crystal Diffraction Intensities, *Crystallography journal*. 1, 3-58.

Boivin, S., Cayre, F.,Herard, J.M., (2000), A finite volume method to solve the Navier–Stokes equations for incompressible flows on unstructured meshes, *International Journal Thermo*,39, 806-825.

Bryan, K. (1969). A Numerical Method for the Study of the Circulation of the World Ocean .*Journal of computational Physics*.4, 347-376

Bryan, K. and Michael, D., (1972). The Circulation Of The World Ocean: A Numerical Study, *Journal of Physical Oceanography*. 2, 319-335

Calhoun, D., LeVegue, R.J., 1999,A Cartesian Grid Finite Volume Method for the Advection-Diffusion Equation in Irregular Geometries.

Casulli, V. and Cattani,E.( 1994).Stability, Accuracy and Efficiency of a Semi-Implicit Method for Three-Dimensional Shallow Water Flow. *Computer Mathematics Applied*. 27, 99-112

Chen ,J.X. and Da Vitoria Lobot, N.,(1995) Toward interactive-rate simulation of Fluids with moving obstacles Using Navier-Stokes Equation , *Graphical Model and image Processing*. 57, 107-116

Clint N. Dawson, Qiang Du, and Todd F. Dupont,Computers, *Elsevier Science Ltd*. 1995, 569-577.

Dawson, C.N., Du,Q., Dupont,T.,F.,(1991). A Finite Difference Domain Decomposition Algorithm for Numerical Solution of the Heat Equation, *Mathematics of Computation*. 57, 63-71.

Dede, M.E., (2000), Multiphysics Topology Optimization of Heat Transfer and Fluid Flow Systems, Proceedings of the COMSOL Conference 2009 Boston.

Diaz-Viera, M.A., Lopez, D.A, Bertchier, A.M., Tapia, O. A, (2008), COMSOL Implementation of a Multiphase Fluid Flow Model in Porous Media, Proceedings of the COMSOL Conference 2008 Boston.

Elfying, T., 1980, Block –Iterative Methods for Consisntent And Inconsistent Linear Equations, *Numerische Mathematik,* 35, 1-12

Estuaries and its Application to the Rotterdam Waterway, *geophysics journal of Astronout Society*, 40, 1-21.

Eymard, R., Gallou¨, T., Herbin, R.,(2006) Finite Volume Methods, Numerical analysis, 7, 713-1020.

Fang, H., He, G., Chen, M.,Liu, X., 2009, Two-Dimensional Numerical Simulation Of Flow And Heat Transport, 10th International Conference on Fluid Control, Measurement and Visualization.

Feldheim, V., Lybaert, P., 2007, Solution Of Radiative Heat Transfer In 2-D Geometries By A Modified Finite-Volume Method Based On A Cell Vertex Scheme Using Unstructured Triangular Meshes, *Numerical Heat Transfer* , 51, 97-119.

Ferziger, J.H., Peric'.,M., (1996). Computational Method for Fluid Dynamics. Berlin Publisher, Third edition.

Foster, N., and Metaxas, D. (1996). Realistic Animation of Liquids, Center for Human Modeling and Simulation,University of Pennsylvania, Philadelphia,1-24.

Gordon, C., Cooper ,C., . Senior, C. A., Banks,H., Gregory, J. M., Johns, T. C., Mitchell, J. F. B., and Wood ,R. A., ( 2000). The Simulation Of SST, Sea Ice Extents And Ocean Heat Transports In A Version Of The Hadley Centre Coupled Model Without Flux Adjustments, *Climate Dynamics*. 16,147-168.

Gordon, C., Cooper, C., Senior, A., Banks, H., Gregory,J.M., Johns, T.C., Mitchell, J.F.B., Wood, R.A., (2000). The Simulation of SST, Sea Ice Extensions and Ocean Heat Transports in a Version of the Hadley Centre Coupled Model without Flux Adjustments, *Climate Dynamics*. 16, 147-168.

Huppert, H. E., (1984). Double – Diffusive Convection Due to Crystallization in Magmas. *Department of Applied Mathematics and theoretical Physics, University of Cambridge*. 12, 11-37.

Keffer, D., (1999). Application and Solution of the Heat Equation in One- and Two-Dimensional Systems Using Numerical Methods, phd(thesis)

Lei,C., Cheng, L., and Kavanagh ,K.( 2000). A Finite Difference Solution Of The Shear Flow Over A Circular Cylinder, *Ocean Engineering*. 27, 271-290

Li ,Z. and Wang, C.,(2002) A Fast Finite Difference Method for Solving Navier-Stokes Equation on irregular Domains.1-15.

Li, Q., Ito, K., Wu, Z., Lowry, S. C., Loheide, S.,T., (2009), Comsol Multiphysics: A Novel Approach to Ground Water Modelling, *Ground water Software Spotlight.* 47, 480-501.

Lindfield, G.R., and Penny, J.E.T.,(2012).*Numerical Method using MATLAB.*Third Edition. Academic Press.

Mitchel, I.A.,( 2007), The Flexible, Extensible And Efficient Toolbox Of Level Set Method, *Journal Science Computer,* 35,300-329.

Murthy, J.Y., Matrhur, S.R., 1998, Finite Volume Method For Radiative Heat Transfer Using Unstructured Meshes, *Journal Of Thermophisycs And Heat Transfer,* 12, 313-356.

Navier-Stokes equations by Krylov subspace and multigrid methods, *Advances in Computational Mathematics*, 4, 27-49. Navier–Stokes Equations, *International Journal For Numerical Methods In Fluids.*44,297-312.

Niceno, B., (2006) A Three Dimensional, Finite Volume Method For Incompressible Navier Stokes Equations On Unstructured, Staggered Grids, *European Conference on Computational Fluid Dynamics,* 1-20.

Powell, A., (2002). Finite Difference Solution of the Heat Equation, *Computational Mathematics,* 1, 1-6.

Puschner, P., Koza, CH., (1989), calculating the maximum execuation time of real –time programs. The journal of the real life system. 1, 159-176.

R.,Berry,M.,Chan,T.F.,Demmel,J.,Donato,J.,Dongarra,J.,Eijkhout,V.,Pozo,R.,Romine,C.,and Van der Vors,H.,(1994).*Templates for the Solution of Linear Systems:Building Blocks for Iterative Methods.*First Edition. Society for Industrial and Applied Mathematics.

Roger, A. and Asce,M.( 1984). Temperature Distributions in Tidal Flow Field, *Journal of Environmental Engineer*. 110, 1099-1116

Roland, A.( 1994), A Model of Wind-and Buoyancy-driven Ocean Circulation, *Journal of Physical Oceanography*.25, 918-941

Salkuyeh, D.K.,(2006). Generalized Jacobi and Gauss-Seidel Methods for Solving Linear System of Equations, *A Journal of Chinese Universities*.16, 164-170.

Servetto, S.D., (2000), Multiple Description Wavelet Based Image Coding, *IEEE Transactions on Image Processing,* 9, 803-825.

Shwartz, P., Barad , M., Colella, P., Ligocki, T., 2005, A Cartesian grid embedded boundary method for the heat equation and Poisson's equation in three dimensions , *Journal of Computational Physics*, 211, 531-550.

Sujanarko, B., Ashari, M., Purnomo, M.H., (2010) Universal Algorithm Control for Asymmetric Cascaded Multilevel Inverter, *International Journal Of Computer Science*, 10, 38-44.

Tachim ,T. and Temam ,R.( 2008). A Two-Grid Finite Difference Method for the Primitive Equations of the Ocean, *ScienceDirect*. 69, 1034-1056

Tartar, L., (2000). An introduction to Navier-Stokes equation and Oceanography. Mellon University, Springer, First Edition.

Thermohaline Convection in a Two-Dimensional Fluid Model and a Comparison with Low-Order Box Models, *Geophysical Astrophysics Fluid Dynamics*, 64, 67-95.

Vatti,V.B.K., and Eneyew,T.K.,(2011). A Refinement of Gauss-Seidel Method for Solving of Linear System of Equations, *International Journal Mathematics and Science*.6, 117-121.

Younes, A., (2003), on modeling the multidimensional coupled fluid flow and heat or mass transportin porous media, *International Journal Of Heat And Transfer*. 46, 367-379.

Young, D.M.,(1977).*Iterative Solution of the Large linear*. First Edition.Dover Publications.

Zhang, K. K. Q., Shotorban, B., Minkowycz W. J., and Mashayek, F.,(2006). A Compact Finite Difference Method On Staggered Grid For Navier–Stokes Flows, *International Journal For Numerical Methods In Fluids*.52,867-881.

Zhou, H., Hu, Z., Zhang, Y., Li, D., 2013, Numerical Implementation for the Filling and Packaging Simulation, John Willing & sons Inc.

Nor Aziran Bt Awang (2012). *Vim And Fdm For Solving Boltzmann Equation Of Self Gravitating Gas-Dust System*. Master's Degree, Universiti Teknologi Malaysia, Skudai.

Benkhaldoun, F., Mahamane, A., Sea¨ıd, Mohammed., 2009, Time stepping schemes for the finite volume solution of diffusion problems, *Elsevier Science,* 1-14.

Tryggvason,G., Bunner, B., Esmaeeli, A., Juric, D., Al-Rawahi, N., 2001, A Front Tracking Method for the Computations of Multiphase Flow, *Journal of Computational Physics,* 169,708-759.