# BROYDEN'S AND THOMAS' METHODS FOR IDENTIFYING SINGULAR ROOTS IN NONLINER SYSTEMS

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For my beloved mother and father,

Zaleha Maikon & Amir Hajib

My little sister and brothers,

My friends,

Raja Nadiah Raja Mohd Nazir

Nurfarhana Osman

Wan Khadijah Wan Sulaiman

and

Muhammad Nurhazrin Mohammad Nawi

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**ABSTRACT**

Nonlinear systems is one of the mathematical models that is commonly used in the engineering and science fields and it is quite complicated to determine the root especially when the problem is singular. This study is conducted in order to study the performance of Broyden's and Thomas' method, which are parts of Quasi-Newton method in solving singular nonlinear systems. By applying the algorithm of each methods, we conduct the calculation to achieve the approximate solutions. MATLAB software is used to compute and present the solutions. Some of useful test problems would describe the properties and usage of the methods. Hence, both methods that have been considered in this study give well approximate solution but Thomas' method gives better results than Broyden's method.

# ABSTRAK

Sistem tak lelurus adalah salah satu model matematik yang biasa digunakan dalam bidang kejuruteraan dan sains dan ia agak rumit untuk mencari penyelesaian lebih-lebih lagi apabila singular berlaku. Kajian ini dijalankan untuk mengkaji prestasi Broyden dan kaedah Thomas yang merupakan sebahagian daripada Kaedah Kuasi-Newton dalam menyelesaikan sistem linear tunggal. Dengan menggunakan algoritma setiap kaedah, kita melakukan pengiraan untuk mendapatkan penyelesaian yang hampir. Perisian MATLAB juga digunakan untuk mengira dan membentangkan penyelesaian. Beberapa contoh masalah dapat menggambarkan sifat dan penggunaan kaedah ini. Oleh itu, kedua-dua kaedah yang telah dipertimbangkan dalam kajian ini memberikan penyelesaian tetapi kaedah Thomas memberikan hasil yang lebih baik daripada kaedah Broyden.

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## LIST OF SYMBOLS

| | | |
|---|---|---|
| $f$ | - | function of equation |
| $x$ | - | variable |
| $f(x)$ | - | function of $x$ |
| $x_0$ | - | initial point |
| $x^*$ | - | local solution |
| $d_k$ | - | search direction in the $k^{th}$ iteration |
| $<$ | - | less than |
| $>$ | - | greater than |
| $=$ | - | equality |
| $\leq$ | - | less than or equal to |
| $\geq$ | - | greater than or equal to |
| $\approx$ | - | approximation |
| $\delta$ | - | limit value of norm |
| $\varepsilon$ | - | epsilon, represents a very small number, near zero |
| $\infty$ | - | infinity symbol |
| $\gamma$ | - | Euler-Mascheroni constant. $\gamma = 0.527721566 \ldots$ |
| [ ] | - | matrix of numbers |
| $\|x\|$ | - | absolute value |
| $\|\|x\|\|$ | - | norm |
| $A^T$ | - | matrix transpose |
| $A^{-1}$ | - | inverse matrix |

| | | |
|---|---|---|
| $rank\ (A)$ | - | rank of matrix A |
| $\dim\ (U)$ | - | dimension of matrix A |
| $\mathbb{R}$ | - | real numbers set |
| $\lim_{x\to\infty} f(x)$ | - | limit value of a function |
| $\sum$ | - | the summation of |

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

Generally, linear systems can be described as the system that the output is proportional to its input which is definitely contradic with a nonlinear systems. A system is said to be nonlinear if it does not contain a linear system where it does not satisfy the superposition principle and its output is not directly proportional to its input. Nonlinear problem also arise in engineering, biology, physic and finance field. In real world problem, most physical systems are inherently nonlinear, such as Navier-Stokes equations in fluid dynamics, Lotka-Volterra in biology and Black-Scholes Partial Differential Equation (PDE) in finance area. A nonlinear system includes any problem that the variables need to be solved but cannot be presented as a linear combination of independent components. Nonlinear equation is quite complicated to solve. Infeasibility to combine the solutions to create new solutions is one of the difficulties in solving nonlinear problems.

Nonlinear equations can be written as $F(x) = 0$ where $F : R^n \to R^n$ is nonlinear mapping. Consider there exists a solution $x^* \epsilon R^n$ . If $F'(x^*)$ is a singular matrix then the nonlinear equations is singular and $x^*$ is a singular root at singular point. Singular root or singular point is said to be the solution, though it is not unique since there are many solution in the range that fulfill the condition of the equations. To understand about singularity, Sánchez (1979) has shown the solution of second-order equation that generally can be written as follows,

$$\omega(z) = (z - z_0)^r \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Based on the solution given, we can say it has singular point if $z = z_0$ and the singular point $z = z_0$ can be classified as regular singular point if function $p(z)$ and $q(z)$ in the equation have at most a pole of order 1 and 2 respectively at $z = z_0$. Since $z = z_0$, therefore $z_0$ is a regular singular point of this equation, then the solution is presented as

$$\omega(z) = (z - z_0)^r \left[ 1 + \sum_{n=0}^{\infty} a_n (z - z_0)^n \right].$$

This is valid when $0 < |z - z_0| < a$, where $a$ represents any maximum value that fulfill the condition of the solution. The expression showed that the solution cannot be at single point as long as the singular points $z - z_0$ have no single value. The points have any other value in the range between 0 and $a$.

From previous discussion, the singular point obtained by the first derivative of nonlinear equations, $F'(x^*)$ is a singular matrix where $x^*$ is a singular root. The singularity of $F'(x^*)$ has potential to determine the convergence behavior of an iterative sequence. Therefore, we consider that $F'(x)$ to be singular on the $S = \{x \in \mathbb{R}^n | \det F'(x) = 0\}$ and $x^* \epsilon S$ if it satisfied the singular assumptions follows ( Buhmiler, 2010) :

i. $F$ is twice Lipschitz continuously differentiable.

ii. Rank $\left(F'(x^*)\right) = n - 1$.

iii. Let $N$ be the null space of $F'(x^*)$ spanned by $\varphi \epsilon \mathbb{R}^n$ and $X$ the range space such that $\mathbb{R}^n = N \oplus X$. For any projection $P_N$ onto $N$ parallel to $X$ we assume

$$P_N F''(x^*)(\varphi, \varphi) \neq 0.$$

From this information, it is clear that when the problems have singularities, we have difficulties to solve it. There are a lot of methods have been discussed that possible to handle this problems.

## 1.2 Background of the Study

Dennis and Jorge (1977) have mentioned that nonlinear problems in finite dimensions are generally solved by iteration and the known method for attacking this problem is Newton's method. Newton's method for nonlinear equations can be derived by assuming that we have an approximation $x_k$ to $x^*$ and that in a neighbourhood of $x_k$ the linear mapping

$$L_k(x) = F(x_k) + F'(x_k)(x - x_k)$$

is a good approximation to $F$. In this case, better approximation $x_k$ to $x^*$ can be obtained by solving the linear system $L_k(x) = 0$. Thus Newton's method takes an initial approximation $x_0$ to $x^*$, and attempts to improve $x_0$ by the iteration,

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}, \qquad k = 0,1, \ldots \tag{1.1}$$

If $F'(x^*)$ is invertible the Newton sequence (1.1) will converge quadratically to $x^*$ if the initial guess, $x_0$ is sufficiently near $x^*$. However, when $F'(x^*)$ fails to be invertible we will say the point $x^*$ is singular. In this case, the Newton iterates will not converge quadratically to $x^*$. The convergence is to be linear if $x_0$ is chosen not only near $x^*$ but in a special type of region that does not contain any ball about $x^*$. (Kelley and Suresh, 1983).

In addition, Dennis and Jorge (1977) have concluded that when Newton's methods is used to find a root and the derivative is singular at the root, convergence of the Newton sequences is in general linear. They are also mentioned that the disadvantages of Newton's method are that a particular problem may require a very good initial approximation to $x^*$ and $F'(x^*)$ need to determine for each $k$.

Hence, the Quasi-Newton method have been proposed as useful modifications of Newton's method for general nonlinear systems of equations. Quasi-Newton methods have potential benefit in solving these algebraic system.. Because of the good potential of Quasi-Newton in solving nonlinear function, in this study we will use Broyden's and Thomas' methods to get the solution for the singular problems.

## 1.3 Statement of the Problem

This research will embark on a study of Broyden's and Thomas' methods, ability in solving singular nonlinear systems.

## 1.4 Objectives of the Study

This study will be conducted to achieve the objectives as follows:

1.4.1 To code Broyden's and Thomas' algorithms using MATLAB.
1.4.2 To apply the Broyden's and Thomas' methods in solving singular nonlinear systems.
1.4.3 To compare the performance of Broyden's and Thomas' algorithms.
1.4.4 To analyze the results of simulation and determine the efficiency of both methods.

## 1.5 Scope of the Study

This study focuses on solving singular nonlinear systems. Broyden's and Thomas' methods are used to handle this problem by approximation the Jacobian according to the formula considered and then injected into the algorithm. The algorithm for both methods is coded using MATLAB.

## 1.6 Significance of the Study

In solving singular nonlinear systems, it is hard to solve using the classical method. Therefore, the Quasi-Newton methods are presented to solve the singular nonlinear systems. This study will give us better understanding on the ability of using the Quasi-Newton methods to solve singular problems. The Broyden's and Thomas' methods are used due to their good behavior to approximate the Jacobian.

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