THE FIRST ISOMORPHISM THEOREM OF LIE GROUP FOR FUZZY TOPOGRAPHIC TOPOLOGICAL MAPPING

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To my beloved mother, father and family.

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ABSTRACT

Fuzzy Topographic Topological Mapping (short FTTM), is a method that was successfully developed to solve the neuromagnetic inverse problem. The method has been developed successfully by using some of algebraic and topological structures. Furthermore, FTTM was evolved as Lie groups for its components, where all components of FTTM 1 and FTTM 2 were shown as 2- dimensional Lie group.The main purpose of this study is to develop the First Isomorphism Theorem of Lie Group for FTTM and also interpret the physical meaning of the First Isomorphism Theorem of Lie Group for Fuzzy Topographic Topological Mapping.

ABSTRAK

Pemetaan Topologi Topografi Kabur (FTTM)merupakan suatu kaedah yang telah berjaya dibangunkan bagi menyelesaikan masalah songsang neuromagnetik menggunakan beberapa struktur aljabar dan topologi. Tambahan lagi, FTTM telah berkembang sebagai kumpulan Lie bagi komponennya, di mana semua komponen FTTM 1 dan FTTM 2 telah ditunjukkan sebagai kumpulan Lie dua dimensi. Tujuan utama kajian ini ialah untuk membangunkan Teorem Isomorfisma Pertama Kumpulan Lie untuk FTTM dan juga untuk mentafsirkan maksud fizikal bagi Teorem Isomorfisma Pertama Kumpulan Lie untuk Pemetaan Topologi Topografi Kabur.

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LIST OF ABBREVIATIONS

BM	-	Base Magnetic Plane
BMI	-	Base Magnetic Image Plane
FM	-	Fuzzy Magnetic Field
FMI	-	Fuzzy Magnetic Image Field
FTTM	-	Fuzzy Topographic Topological Mapping
FTTM 1	-	Fuzzy Topographic Topological Mapping 1
FTTM 2	-	Fuzzy Topographic Topological Mapping 2
MC	-	Magnetic Contour Plane
MCI	-	Magnetic Image Plane
TM	-	Topographic Magnetic Field
TMI	-	Topographic Magnetic Image plane

LIST OF FIGURES

LIST OF SYMBOLS

$f \circ g$	-	Composition of mapping *f* and *g*
$\in$	-	Element of
$\forall$	-	For All
$\cong$	-	Homeomorphism/ Isomorphism
$f: X \longrightarrow Y$	-	Mapping *f* from X to Y
Max	-	Maximum
Min	-	Minimum
$\therefore$	-	Therefore

CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

Isomorphism theorems are theorems that describe the relationship between quotient, homomorphism and sub objects. Versions of the theorem exist for group, ring, vector space, module, lie algebra and various algebraic structures.

In this chapter we are going to introduce some of the concepts that are related to isomorphism theorem which are group, quotient group, homomorphism, kernel of group and finally Lie group.

1.1.1 Group

The term group was used by Galois around 1830 to describe sets of one-to-one functions on finite sets that could be grouped together to form a closed set. As is the case with most fundamental concepts in mathematics, the modern definition of a group that follows is the result of a long evolutionary process. Although this definition was given by both Heinrich Weber and Walter Von Dyck in 1982, it did not gain universal acceptance until the twentieth century.

In other words, a group is a set together with an associative operation such that there is an identity, every element has an inverse, and any pair of elements can be combined without going outside the set [2].

Definition 1.1 [2]

Let G be a group and H a normal subgroup of G. A quotient group is the set $G/H = \{aH \mid a \in G\}$ such that $(aH)(bH) = abH$.

The resulting quotient is written G/H where G is the original group and H is the normal subgroup of G.

Example 1.1

Let $G = \mathbb{Z}_8$ and let $H = \langle 2 \rangle = \{0, 2, 4, 6\}$, Then

$G/H = \{0+H, 1+H\}$

1.1.2 Homomorphism Group [4].

Let G be a group with respect to $\boxplus$ and let H be a group with respect to $\otimes$. A homomorphism from G to H is a mapping $\psi : G \longrightarrow H$ such that $\psi(x \boxplus y) = \psi(x) \otimes \psi(y)$, $\forall\, x, y$ in G , therefore, G is homeomorphic to H.

1.1.3 Types of Homomorphisms

There are different kinds of homomorphisms and some special homomorphisms have special names

If the homomorphism $\psi: G \longrightarrow H$ is injective, we say that it is a monomorphism, and if ψ is surjective we call it an epimorphism. When it is both injective and surjective (that is bijective) it is called an isomorphism.

In the later case we also say that G and H are isomorphic, meaning they are basically the same group (have the same structure). A homomorphism from G to itself is called an endomorphism, and if it is bijective then it is called an automorphism [7].

Definition 1.3 [4]

Let ϕ be a homomorphism from the group G to the group H. The Kernel of ϕ is the set Ker $\phi = \{x \in \mathrm{G} \mid \phi(x) = \bar{e}\}$, where $\bar{e}$ denotes the identity element in H.

1.1.4 Group Isomorphism

Two groups G and H are called isomorphic if there is a bijection map $\varphi : G \longrightarrow H$ such that $\forall\, x, y$ in G, $\varphi(xy) = \varphi(x)\, \varphi(y)$. If there exists an isomorphism between groups, they are termed isomorphic groups [2].

1.1.5 The First Isomorphism Theorem

The first isomorphism theorem states that the image of any group G under a homomorphism is always isomorphic to a quotient of G, specially, the image of G under a homomorphism $\psi : G \longrightarrow H$ is isomorphic to $G/Ker(\psi)$ where $ker(\psi)$ denotes the kernel of ψ [14]. Formally, it is given in the following theorem.

Theorem 1.1 [4] (First Isomorphism Theorem)

If ψ is an epimorphism from the group G to the group H. Then H is isomorphic to $G/Ker(\psi)$.

$$G/Ker(\psi) \cong \mathrm{Im}(\psi).$$

The following diagram illustrates the meaning of first isomorphism theorem.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d] & H \\ G/Ker(\psi) \arrow[ur] & \end{tikzcd}$$

1.1.6 Lie Group

Only a century has elapsed since 1873, when Sophus Lie began his research on what has evolved into one of the most fruitful branches of modern mathematics- the theory of Lie groups. Few years later lie groups have come to play an increasingly important role in modern physical theories where it enter physics through their finite- and infinite- dimensional matrix representations.

Lie group is situated at the intersection of two basic areas of mathematics: algebra and geometry. A Lie group is a group. It is also a differentiable (smooth) manifold. Finally, the two structures, algebraic and the geometric structures, have to be compatible with each other in a precise manner.

A Lie group is a group of symmetries where these symmetries are continuous. For example, a circle has a continuous group of symmetries where we can rotate the circle randomly [5].

1.2 Influential Observation

Fuzzy Topographic Topological Mapping (FTTM) is a novel method for solving neuromagnetic inverse problem to determine the current source. i.e. epileptic foci. The recorded magnetic fields help in determining where electrical currents originate and the strength of currents. Currently, there are two versions of FTTM namely FTTM 1 and FTTM 2. FTTM Version 1 is designed to present a 3-D view of an unbounded single current source in one angle observation (upper of a head model). It consists of three algorithms, which link between four components of the model as shown in Figure 1.1.

The four components are Magnetic Contour Plane (MC), Base Magnetic Plane(BM), Fuzzy Magnetic Field (FM) and Topographic Magnetic Field (TM) (Figure 1.1). MC is actually a magnetic field on a plane above a current source with z=0. The plane is lowered down to BM, which is a plane of the current source with z= -h. then the entire BM is fuzzified into a fuzzy environment (FM), where all the magnetic field reading are fuzzified. Finally, a three dimensional presentation of FM is plotted on BM. The final process is defuzzification of the fuzzified data obtain a 3-D view of the current source (TM).

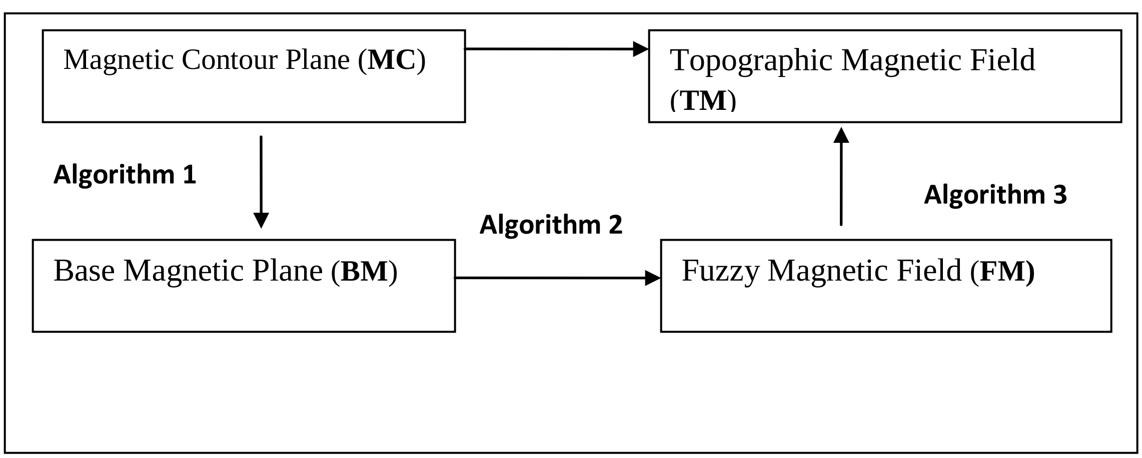

Figure 1.1: FTTM Version 1

FTTM Version 2 has been developed to present 3-D view of a bounded multi current source [12] in 4 angles of observation (upper, left, right and back of a head model). It consists of three algorithms, which link between four components of the model. The four components are Magnetic Image Plane (MI), Base Magnetic Image Plane (BMI), Fuzzy Magnetic Image Field (FMI) and Topographic Magnetic Image Field (TMI) (Figure 1.2).

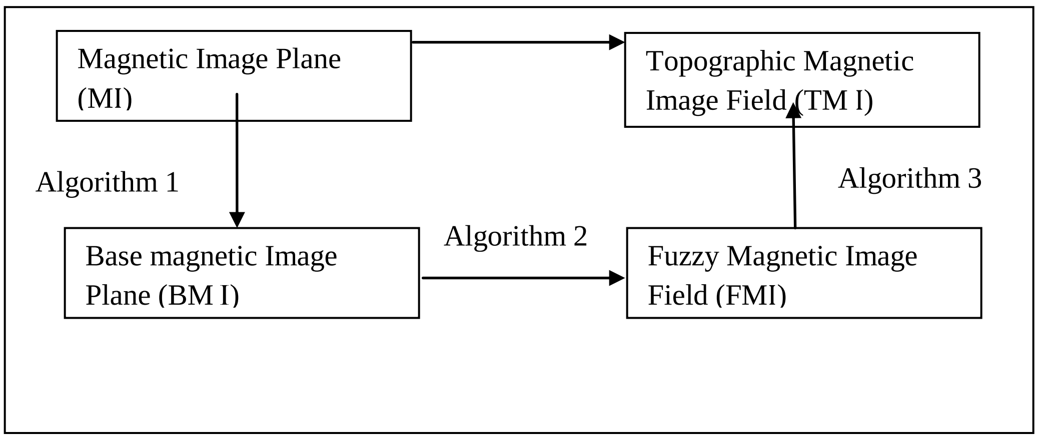

Figure 1.2: FTTM Version 2

MI is a plane above a current source with z=0 containing all grey scale reading (0DN-255DN) of magnetic field. The plane is lowered down to BMI, which is a plane of the current source with z= -h. Then the entire base BMI is fuzzified into fuzzy environment (FMI), where all the gray scale reading are fuzzified. Finally, a three dimensional presentation of FMI is plotted on BMI. The final process is defuzzification of the fuzzified data to obtain a 3-D view of the current source (TMI).

FTTM Version 1 as well as FTTM Version 2 is specially designed to have equivalent topological structure between its components. In other words, a homomorphism between each component of FTTM Version 1 as well as FTTM Version 2 exists. The homeomorphism between a unit sphere (denoted by S^2) and an ellipsoid with $|x| \leq 1$, $|y| \leq 1$ and $|z| \leq 2$ (denoted by E^2) is existed [10]. (see Figure 1.3).

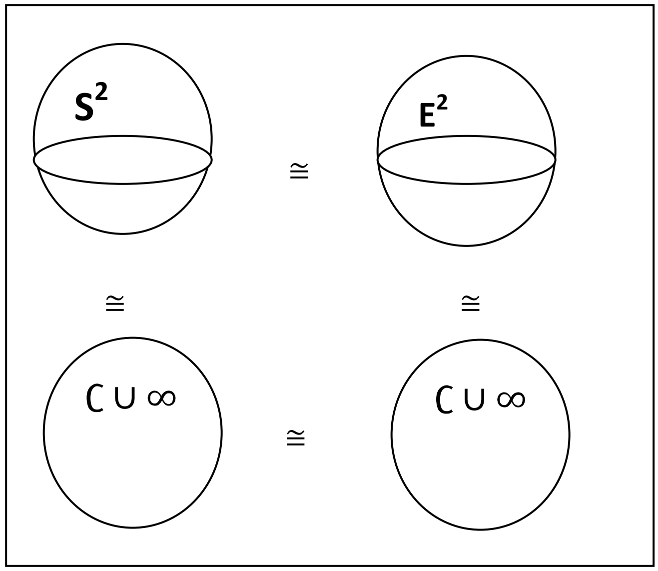

Figure 1.3: Homeomorphism from S^2 to E^2

Magnetic field reading generated by MATLAB programming and FTTM Version 1, was used to solve neuromagnetic inverse problem of MEG for an unbounded single current source. An algorithm was written to determine location, the direction and magnitude of an unbounded single current source [6]. Furthermore, magnetic field readings generated by MATLAB programming and FTTM Version 2, was also used to solve neuromagnetic inverse problem for a bounded multi current sources. In addition, Fuzzy C-means is applied to identify the number of current sources by simulation and experimental [6].

1.3 Problem Statement

The main aim of this research is to prove The First Isomorphism Theorem of Lie group for Fuzzy Topographic Topological Mapping Version 1 and The First Isomorphism Theorem of Lie Group for Fuzzy Topographic Topological Mapping Version 2.

1.4 Objectives of Research

The objectives of this research are given as follows:

(i) To prove The First Isomorphism Theorem of Lie Group of Fuzzy Topographic Topological Mapping Version 1 (FTTM 1).

(ii) To prove The First Isomorphism Theorem of Lie Group of Fuzzy Topographic Topological Mapping Version 2 (FTTM 2).

(iii) To interpret the physical meaning of the developed these theorems.

1.5 Scope of Research

The scope of this research will be on the First Isomorphism Theorem, Lie group and Fuzzy Topographic Topological Mapping (FTTM) Version 1 as well as FTTM Version 2.

1.6 Significance of Research

The main purpose of this research is to develop the First Isomorphism Theorem of Fuzzy Topographic Topological Mapping FTTM whereby we can have

$$\mathrm{MC} \cong \mathrm{MC}/\,\mathrm{Ker}(\boldsymbol{\tau}), \text{ where } \boldsymbol{\tau} \text{ is a function in } bm,\ fm \text{ and } tm.$$

1.7 Dissertation’s Layout

This dissertation contains five chapters which are divided as follows:

Chapter 1 deals with the introduction to the research. It discusses the background and motivation, influential motivation, problem statement, objectives and finally, scope of the research. Then, Chapter 2 presents the literature review of the research. It discusses FTTM of two different forms FTTM 1 and FTTM 2, Lie group of FTTM. It is then followed by Chapter 3, which presents the proof of the First Isomorphism Theorem of Lie group for FTTM 1 and FTTM 2. The physical interpretation of these theorems will be in Chapter 4. Some conclusions and recommendations will be presented base on the results in Chapter 5.

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