THE COMMUTATIVITY DEGREE OF 3-ENGEL GROUPS UP TO ORDER 40

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To my beloved mother and father

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ABSTRACT

Friedrich Engel has discovered Engel groups in the 50s in Germany. The origin of this group lies in the theory of Lie algebras. As an example, one of the basic classical results for Engel Lie algebras is Engel's theorem. In this research, the 3-Engel groups of order up to 40 will be the main focus. An application of 3-Engel groups on probability theory will be presented too. The commutativity degree can also be viewed as the probability that two elements in the group commute denoted by $P(G)$. This probability applied to the eighteen 3-Engel groups with order at most 40 is proven to be at most 5/8. A software named Groups, Algorithms and Programming (GAP) software is used to facilitate most of the calculations in this research.

ABSTRAK

Friedrich Engel telah menemui kumpulan Engel pada awal 50-an di German. Pada asalnya, kumpulan ini terdapat dalam teori algebra Lie. Sebagai contoh, salah satu hasil klasik asas untuk algebra Lie Engel adalah teorem Engel. Dalam disertasi ini, kumpulan 3-Engel pada peringkat kurang kurang dari dan sama dengan 40 menjadi focus utama. Aplikasi kumpulan 3-Engel dalam teori kebarangkalian juga turut dibincangkan. Darjah kekalisan tukar tertib juga dapat dilihat sebagai kebarangkalian bagi dua unsur dalam kumpulan yang kalis tukar tertib diwakili sebagai $P(G)$. Kebarangkalian ini diaplikasikan dalam lapan belas kumpulan 3-Engel dengan peringkat kurang dari dan sama dengan 40 telah dibuktikan kurang daripada 5/8. Sebuah perisian yang dinamakan "*Groups, Algorithms and Programming*" (GAP) digunakan bagi memudahkan sebahagian besar pengiraan dalam kajian ini.

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LIST OF SYMBOLS/ ABBREVIATIONS/ NOTATION

$1, e$	Identity
GAP	Groups, Algorithms and Programminh
$P(G)$	Commutativity degree
$k(G)$	Number of conjugacy classes
G	A finite groups G
$\|G\|$	Order of a finite group G
$[x, y]$	Commutator of x and y, $x^{-1}y^{-1}xy$
$[x, {}_3y]$	$[x, y, y, y] = [[[x, y], y], y]$
$H \lhd G$	H is normal in G
$< a >$	Cyclic group
$cl(a)$	The conjugacy class of a in G
x^y	The conjugate of x in y, $y^{-1}xy$
x^G	The set of all conjugates, $g^{-1}xg, \forall g \in G$
$\in$	Element of
$\neq$	Not equal to
$\forall$	For all
$\rtimes$	Semi direct product
$\blacksquare$	End of proof

CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

Friedrich Engel has discovered Engel groups in the 50's in Germany. The origin of this group lies in the theory of Lie algebras. As an example, one of the basic classical results for Engel Lie algebras is Engel’s theorem.

This research begins on viewing the history of Friedrich Engel and the basic properties of n-Engel groups. In this research, the 3-Engel groups will be our main focus. More characteristics of this group will be discussed and several examples will be presented and used. An application of 3-Engel groups on probability theory will be presented too as again as our main research. A software named Groups, Algorithms and Programming (GAP) software is use to facilitate some of the calculations in this research.

1.2 Research Background

This research begins with an overview of the history of Friedrich Engel. Some basic properties of n-Engel groups are given especially on 3-Engel groups. Some characteristics of this group will be presented with the proofs. These include some definitions, theorems and examples. A software called Groups, Algorithms and Programming (GAP) is used to identify the 3-Engel groups of small order (groups of order up to 40). The commutativity degree of a group can also be viewed as the probability that two random elements in the group commute.

1.3 Problem Statement

Which groups are 3-Engel groups? What type of characteristics do they have? How commutative are they?

1.4 Research Objectives

The main objectives of this research are:

1. to find the characteristics of 3-Engel groups and present the proofs.
2. to use GAP in order to determine small groups (groups up to order 40) that are 3-Engel.
3. to calculate the degree of commutativity degree, $P(G)$ of given 3-Engel groups by using two different approaches;
   (i) Cayley table,
   (ii) Conjugacy classes.

1.5 Scope of Research

This research will focus only on all 3-Engel groups of order up to 40.

1.6 Significance of Study

The result of this research can be used for further research in related areas. Research paper will also be sent to national and international indexed journal such as journal of "Sains Malaysiana", Bulletin of the Malaysian Mathematical Sciences Society, and World Scientific. Furthermore, this research will enhance contribution from mathematicians in Malaysia especially in Pure Mathematics fields and Statistics.

1.7 Research Methodology

This dissertation will be carried out according to the following steps:

1. Study and understand the concepts of 3-Engel groups.
2. Review and prove some given basic properties of 3-Engel groups.
3. Find more characterization of 3-Engel groups and present the proofs.
4. Identify some specific small groups (groups up to order 40) that fulfill the characteristics of 3-Engel.
5. Find the commutativity degree of these 3-Engel groups.
6. Write up of dissertation.
7. Presentation of dissertation.

1.8 Dissertation Report Organization

This dissertation organized into five chapters. Chapter 1 includes the research framework.

In Chapter 2, the history of Friedrich Engel is overviewed and some definitions and basic concepts that will be used throughout the dissertation are introduced. Besides, a simulation model on detecting 3-Engel groups by GAP is included too.

Chapter 3 reviews on some characteristics of 3-Engel groups. Some definitions and theorems involving 3-Engel groups are presented with complete proof. All examples of 3-Engel groups of order at most 40 are included together with some of non examples of 3-Engel group with order less than 40.

Chapter 4 discusses the application of 3-Engel groups in probability theory. This chapter reviews on some theorems involving conjugacy classes, $k(G)$ and the commutativity degree of a group, $P(G)$. The result on the commutativity degree of all 3-Engel groups of order at most 40 with different approaches is presented and analyses.

Lastly, in Chapter 5, the obtaining results are summarized. Suggestions for further research are also included.

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