THE COMMUTATIVITY DEGREE OF ALL NONABELIAN METABELIAN GROUPS OF ORDER AT MOST 24

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To

Umie, aboh and all my siblings

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ABSTRACT

A metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group G/A is abelian. Meanwhile, the commutativity degree can be viewed as the probability that two elements in a group commute, denoted by $P(G)$. The main objective of this research is to compute the commutativity degree of all metabelian groups of order at most 24. Some basic concepts related with $P(G)$ will first be presented. Two approaches have been used to compute $P(G)$, where G is a metabelian group of order at most 24, namely the 0-1 Table and the Conjugacy Class Method. A software named Groups, Algorithms and Programming (GAP) have been used to facilitate the computations of the commutativity degree.

ABSTRAK

Kumpulan metabelan adalah satu kumpulan yang mempunyai subkumpulan komutatornya abelan. Dengan erti kata lain, satu kumpulan G adalah metabelan jika dan hanya jika wujud satu subkumpulan normal yang abelan, A dengan kumpulan faktornya G/A adalah abelan. Sementara itu, darjah kalis tukartertib ditakrifkan sebagai kebarangkalian bahawa dua unsur dalam satu kumpulan adalah kalis tukar tertib, ditandakan sebagai $P(G)$. Objektif utama bagi penyelidikan ini adalah untuk mengira darjah kalis tukartertib kumpulan metabelan dengan peringkat tidak melebihi 24. Dalam penulisan ini, beberapa konsep asas tentang $P(G)$ dibentangkan. Dua pendekatan telah digunakan untuk mengira $P(G)$, G adalah satu kumpulan metabelan dengan peringkat tidak melebihi 24, kaedah dinamakan Jadual 0-1 dan Konjugasi Kelas. Satu Perisian dinamakan Groups, Algorithms and Programming (GAP) telah digunakan untuk mengira darjah kalis tukartertib.

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LIST OF SYMBOLS/ ABBREVIATIONS/ NOTATIONS

$[a,b]$	Commutator of a and b, $a^{-1}b^{-1}ab$
$< a >$	Cylic subgroup generated by a
$\mathrm{cl}(a)$	Conjugacy class of a
$c(G)$	The set of all commutativities is the event “randomly chosen x and y commute”
C_x	The centralizer of x
D_n	The dihedral group of order $2n$
e	Identity
GAP	Groups, Algorithms and Programming Software
G	A finite group G
G/A	Quotient group
$\|G\|$	Order of a finite group G
G'	Commutator subgroup
$H < G$	H is a subgroup G
$H \triangleleft G$	H is normal in G
$k(G)$	The number of conjugacy classes of G
$\mathbb{N}$	Natural numbers
$P(G)$	The probability that two elements commute in G
$Z(G)$	Center of a group G
$\mathbb{Z}$	Integers
$\in$	Element of
$\neq$	Not equal to
$\notin$	Not an element of
$\rtimes$	Semi direct product

□	End of definition/ theorem/ lemma/ proposition
■	End of proof
$\cong$	Isomorphic

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

A metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group G/A is abelian. A metabelian group also is a solvable group of derived length two [1]. In the Russian mathematical literature, by a metabelian group one sometimes means a nilpotent group of nilpotency class two [1].

In the past 20 years, and particularly during the last decade, there has been a growing interest in the use of probability in finite groups. In recent years, the probabilistic methods have been proved to be useful in the solution of several difficult problems in group theory. In some cases the probabilistic nature of the problem has been apparent from its formulation, but in other cases the use of probability seems surprising, and cannot be anticipated by the nature of the problem.

All groups considered in this reseach will be assumed finite. The commutativity degree of a group G, which is denoted by $P(G)$, is the probability that two elements of the group G, chosen randomly with replacement, commute. This can be written as,

$$P(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G}$$

$$= \frac{\left|\{(x, y) \in G \times G \,|\, xy = yx\}\right|}{|G|^2}.$$

Therefore in this research, all concepts on the probability that two elements commute are applied and computed for nonabelian metabelian groups. The characteristics of these groups are explored too. A software named Groups, Algorithms and Programming (GAP) has been used to verify some of the results found in this research [2].

1.2 Statement of the Problem

What are metabelian groups? What are their commutativity degrees ?

1.3 Objectives of the Study

The research objectives are :

1) to study on all metabelian groups of order at most 24,
2) to find the conjugacy class of all nonabelian metabelian groups of order at most 24,
3) to find the Cayley Table and the 0-1 Table of all nonabelian metabelian groups of order at most 24, and
4) to find the commutativity degree of all nonabelian metabelian groups of order at most 24.

1.4 Scope of the Study

This research will focus on all metabelian groups of order at most 24.

1.5 Significance of the Study

This research gives advantages in mathematics education as it extends the knowledge in Pure Mathematics areas especially in group theory. The findings of this research can be beneficial as a reference for other researchers in gaining new findings in mathematics.

1.6 Summary of Each Chapter

This dissertation is organized into six chapters. Chapter 1 gives a short introduction on metabelian groups and commutativity degree. This chapter also includes Background of the Problem, Statement of the Problem, Objectives of the Study, Scope of the Study and Significance of the Study.

Chapter 2 includes an overview on the metabelian groups. This chapter focuses on details about metabelian groups and introduces some basic concepts and properties on metabelian groups that will be used in the subsequent chapters. These include some definitions, propositions, theorems and examples that are related in the determination of metabelian groups.

Chapter 3 includes the commutativity degree of groups. This chapter focuses on researches done by different authors on probability, also known as the commutativity degree. Some definitions, propositions, theorems and proofs that are related to the commutativity degree are also included.

In Chapter 4, the application of metabelian groups in probability theory is discussed. The result on the commutativity degree of all metabelian groups of order less than 24 will be represented. There are two methods to find the commutativity degree of groups which are using Cayley Table or 0-1 Table, and Conjugacy Classes. The Groups, Algorithms and Programming (GAP) software have been used to facilitate in finding the Cayley Table for some groups.

In Chapter 5, the commutavity degree of all metabelian groups of order 24 are represented. The Groups, Algorithms and Programming (GAP) software have been used to facilitate in finding the Cayley Table for these groups.

Finally, in Chapter 6, the obtained results are summarized. Suggestions for further research are also given in this chapter.

REFERENCES

1. Kurosh, A.G.,*The theory of groups* , 1–2 , Chelsea (1955–1956) (Translated from Russian). 1955.
2. Rainbolt, J.G. and Gallian, J.A., *A Manual to be Used with Contemporary Abstract Algebra*. 5th Edition. Houghton Mifflin Company. 2003.
3. Rotman, J. J., *An Introduction to the Theory of Groups.* 4th Edition. New York: Springer-Verlag, Inc. 1994.
4. Rose, J. S., *A Course on Group Theory.* New York: Dover. 1994.
5. Schmidt, O.U., *Abstract Theory of Groups.* USA: W.H Freeman and Company. 1966.
6. Robinson, D. J. S., *A Course in the Theory of Groups*. New York: Springer Verlag. 1993.
7. Redfield, R. H., *Abstract Algebra A Concrete Introduction*. Addison Wesly. Longman, Inc.2001.
8. Fraleigh, J.B., *A First Course in Abstract Algebra.* 7th Edition. USA: Addison Wesly.Longman, Inc.2000.
9. Rainbolt, J.G., *A First Course in Abstract Algebra*. 2nd Edition. Prentice Hall. Inc.2000.
10. Abdul Rahman, S.F., Metabelian *Groups of Order at Most 24.* Msc.Dissertation, Universiti Teknologi Malaysia, November 2010.
11. Dixon, J.D., *Probabilistic Group Theory.* Carleton University. 2004.
12. Belcastro, S.M., and Sherman, G.J., *Counting Centralizers in Finite Groups*. *Mathematics Magazine*. 67(5): 366-373. 1994.

13. Erdos, P. and Turan, P., “*On some problems of statistical group theory*.” *Acta Math.Acad. of Sci. Hung*, **19**: 413-435. 1968.

14. Gustafson,W., “*What is the probability that two group elem*ents *commute*?”American Mathematical Monthly, **80**: 1031-1034. 1973.

15. MacHale, D., *How Commutative Can a Non-Commutative Group Be*? *The Mathematical Gazette.* 58: 199-202. 1974.

16. Rusin, D.J., *What is the probability that two elements of a finite group commute*? Pacific Journal of Mathematics, 82(1): 237-247. 1979.

17. Lescot, P., *Central extensions and commutativity degree*, Comm. Algebra 29, no. 10, 4451– 4460. 2001.

18. Blackburn, S., Groups of Prime Power Order with derived subgroup of prime Order. J. Algebra. 625-627. 1999.

19. Castalez , A., *Commutativity Degree of Finite Groups*. Msc.Dissertation, Wake Forest University, Winston-Salem,North Carolina. May 2010.

20. Barry, F., MacHale, D., and N´i Sh´e, A., *Some supersolvability conditions for finite groups*, Math. Proc. Royal Irish Acad. 106A (2). 163–177. 2006.

21. Guralnick, R.M., and Robinson, G. R., *On the commuting probability in finite groups*, J. Algebra 300 (2) . 509–528. 2006.

22. Pournaki, M.R., and Sobhani, R., *Probability that the commutator of two group elements is equal to a given elements*, J. Pure Appl. Algebra 212. 727-734. 2008.

23. Funk, D., *Are some finite groups less abelian than others* ? Senior Project Paper. 2009.