AUTOMORPHISM GROUPS OF METACYCLIC GROUPS OF CLASS TWO

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I would like to dedicate this work to my following beloved ones:

First of all, to the only man in my life, my husband who provided me with all of his love, efforts and support. Second, to the candles of my life and the reason of my happiness, my handsome boys, Mohamed and Abdualmohaimen. Third, to the function of love, passion, sacrifice and bless; my parents. Finally, I dedicate my work to all the UTM lecturers.

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ABSTRACT

An automorphism of a group G is an isomorphism from G to G, which is one to one, onto and preserving operation. The automorphism of G forms a group under composition, and is denoted as $Aut(G)$. A group is metacyclic if there is a normal cyclic subgroup whose quotient group is also cyclic. In 1973, King classified metacyclic $p-$group while in 1987, Newman developed a new approach to metacyclic $p-$groups suggested by the $p-$group generation algorithm. They found new presentation for these groups. The automorphism groups can be separated to inner and outer automorphisms. An inner automorphism is an automorphism corresponding to conjugation by some element a. The set of all automorphisms form a normal subgroup of $Aut(G)$. The automorphism group which is not inner is called outer automorphism and denoted as $Out(G)$. In this research, automorphism groups of split and non-split metacyclic groups of class two will be investigated including the inner and outer automorphisms.

ABSTRAK

Suatu automorfisma bagi suatu kumpulan G ialah suatu isomorfisma dari G ke G, iaitu satu ke satu, keseluruh dan mengekalkan operasi. Automorfisma G membentuk suatu kumpulan di bawah gubahan dan ditandakan sebagai $Aut(G)$. Suatu kumpulan dikatakan meta-kitaran jika wujud sub-kumpulan kitaran yang normal dengan kumpulan nisbah tersebut juga berkitar. Pada tahun 1973, King mengklasifikasikan kumpulan $-p$ meta-kitaran manakala pada tahun 1987, Newman membangunkan suatu pendekatan baru untuk kumpulan $-p$ meta-kitaran; dicadangkan oleh algoritma generasi kumpulan $-p$. Mereka telah menemui persembahan baru bagi kumpulan-kumpulan tersebut. Kumpulan-kumpulan automorfisma boleh dibahagikan kepada automorfisma dalaman dan automorfisma luaran. Satu automorfisma dalaman adalah automorfisma sepadan dengan konjugat suatu unsur a. Semua set automorfisma membentuk satu sub-kumpulan normal bagi $Aut(G)$. Automorfisma yang bukan dalaman dipanggil automorfisma luaran dan ditandakan sebagai $Out(G)$. Dalam kajian ini, kumpulan automorfisma bagi kumpulan meta-kitaran kelas dua yang dikaji termasuk automorfisma dalaman dan luaran.

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LIST OF SYMBOLS

$Aut\ (G)$	Automorphism group
$Inn\ (G)$	Inner automorphism
$\langle a \rangle$	Generator of a group
$Z(G)$	Center of a group
$Out(G)$	Outer automorphism
$G/_{A}$	Factor group
$\lhd$	Normal group
$G = H \rtimes N$	Semi-direct product
$Hom(K, H)$	Homomorphism from K to H
$C_H(K)$	Conjugation with H in K
$C_K(H)$	Conjugation with K in H
$G = HK$	Non-split group
$\lvert G \rvert$	Order of a group
$Im\ \beta$	Image β
$\subseteq$	Subset
$CrossHom(K, H)$	Crossed homomorphism from K to H
$[x, y]$	The commutator of x and y

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

In mathematics, an automorphism is an isomorphism from a mathematical object to itself. The set of all automorphisms of an object forms a group is called an automorphism group, which is also the symmetry group of the object. In category theory, an endomorphism (that isomorphism from an object to itself) and an isomorphism is called an automorphism. A morphism is an abstraction derived from structure-preserving mappings between two mathematical structures, so if a morphism f has domain X and codomain Y we write $f : X \to Y$. Thus a morphism is represented by an arrow from its domain to its codomain. In set theory, morphisms are functions. A function assigns exactly one output to each input. For example, a function might associate the letter A with the number 1, the letter B with the number 2, and so on. One precise definition of a function is an ordered triple of sets, written (X, Y, F) where X is the domain, Y is the codomain, and F is a set of ordered pairs (a, b). In set theory especially, a function f is often defined as a set of ordered pairs, with the property that if (x, a) and (x, b) are in f, then $a = b$. So morphisms are not necessarily functions and objects are not necessarily sets. However, the objects will be sets with some additional structure and the morphisms will be functions preserving that structure [1].

In 1856, an Irish mathematician Willian Rowan Hamilton discovered one of the earliest group automorphism (automorphism of a group, not simply a group of automorphism of points). Furthermore, he discovered an order two automorphism in his Icosian Calculus. In some contexts, the trivial automorphism is the identity a morphism, other (non-identity) automorphism is nontrivial automorphisms [2].

In Riemannian geometry, an automorphism is a self-isometry which is called the isometry group. An automorphism of a set X is an arbitrary permutation of the elements of X. It is also called the symmetric group on X. Formally, it is a permutation of the group elements such that the structure remains unchanged. There is a natural group homomorphism $G \rightarrow Aut(G)$ whose image is the group inner automorphism, $Inn(G)$ and the kernel is the center of G for every group G. Thus if G has a trivial center it can be embedded into its own automorphism group [2].

In the category of Riemann surfaces, an automorphism is an objective biholomorphic map from a surface to itself. For example, the automorphism of a Riemann sphere is a Mobiu's transformation. Usually, a nice description of an automorphism group is possible because of the presence of a coordinate system (normal form of elements), or it can be explained by a connection with linear groups (topological or geometrical objects) or in another way [3].

In most cases, familiar groups have arisen naturally in the form of automorphism groups of some mathematical and nonmathematical structures. During recent years, the interest in studying automorphism groups has grown there is no exception to the groups themselves. In 1894, Richard Dedekind developed the idea of an automorphism of a field, which he called permutation of a field. The earlier applications of group theory to the theory of equations were through groups of permutations of the roots of certain polynomials [4].

In 1895, Heinrich Weber continued Dedekind’s approach to groups acting on fields in his algebra text. It was not until the 1920s, after Emmy Noether’s abstract approach to algebra became influential in Gottingen, that Emil Artin (1898-1962) developed this relationship of groups and fields in great detail. He also emphasized that the goal of what is now called Galois theory should not be to determine solvability conditions for algebraic equations, but to explore the relationship between field extensions and groups of automorphisms [4].

1.2 Statement of the Problem

What are the automorphism groups of split and non-split metacyclic groups of class two?

1.3 Objectives of the Study

The objectives of this study are:

1. to study on automorphism groups, including all the properties, characteristics and types,
2. to study on metacyclic groups, including their properties, theorems and examples,
3. to determine the automorphisms of split and non-split metacyclic groups of class two.

1.4 Scope of the Study

This research will focus on inner and outer automorphisms of metacyclic groups of class two.

1.5 Significance of the Study

This research should lead to explore automorphism groups of metacyclic groups of class two. The automorphism groups of metacyclic groups of class two have not been found by any researcher. Thus this research will produce new results. This is also true for inner, non-inner and outer automorphisms of groups of class two.

1.6 Dissertation Report Organization

Chapter 1 describes an introduction to the whole thesis. This chapter introduces the concepts of the automorphism groups of metacyclic groups of class two. This chapter also includes the background of the research, statement of problem, objectives of the study, scope of the study and significance of the study.

Chapter 2 states some basic definitions and concepts in group theory that will be used in the following chapters. Before we proceed to the chapter of the automorphism group of non-split metacyclic *p*-group of class two, some basic results of the automorphism groups, nilpotent group of class two and metacyclic groups are included.

In Chapter 3, some results of automorphism groups of non-split metacyclic *p*-groups of class two including theorems, lemmas and examples are presented. Meanwhile, the automorphism groups of split metacyclic *p*-groups of class two are stated in Chapter 4.

Finally, Chapter 5 includes the conclusion for this research and suggestion for further research.

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