INTEGRAL EQUATION APPROACH FOR COMPUTING GREEN'S FUNCTION ON UNBOUNDED SIMPLY CONNECTED REGION

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ABSTRACT

This research is to compute the Green's function on an unbounded simply connected region by conformal mapping and by solving an exterior Dirichlet problem. The exact Green's function is found by using Riemann mapping and Möbius transform. The Dirichlet problem is then solved using a uniquely solvable Fredholm integral equation on the boundary of the region. The kernel of this integral equation is the generalized Neumann kernel. The method for solving this integral equation is by using the Nyström method with the trapezoidal rule to discretize it into a system. The linear system is solved by the Gaussian elimination method. As an examination of the proposed method, several numerical examples for some various test regions are presented. These examples include a comparison between the numerical result and the exact solutions.

ABSTRAK

Kajian ini adalah untuk mengira fungsi Green pada rantau terkait ringkas tak terbatas dengan kaedah pemetaan konformal dan dengan kaedah menyelesaikan masalah Dirichlet luaran. Fungsi tepat Green boleh ditemui dengan menggunakan pemetaan Riemann dan penjelmaan Möbius. Masalah Dirichlet kemudiannya diselesaikan menggunakan persamaan kamiran Fredholm yang mempunya payeleoaian unik di sempadan rantau ini. Inti persamaan kamiran ini adalah inti Neumann teritlak. Kaedah untuk menyelesaikan persamaan kamiran ini ialah dengan menggunakan kaedah Nyström dengan petua trapezoid untuk diskritkannya kepada sebuah sistem. Sistem linear diselesaikan dengan kaedah penghapusan Gauss. Untuk mengkaji kaedah ya ng dicadangkan, contoh-contoh berangka bagi beberapa rantau ujian dibentangkan. Contoh-contoh ini termasuk perbandingan antara keputusan berangka dan penyelesaian tepat.

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CHAPRER 1

RESEARCH FRAMEWORK

George Green (1793–1841)

1.1 Introduction

Green's functions are presented by the British mathematician George Green (1793-1841), who first developed this concept in 1830s. In the modern linear partial differential equations, Green's functions are analyzed largely from the point of view of fundamental solutions. In the following sections, Green's functions are described in one-dimension and two-dimension space.

Green's function in one-dimension

Green's function in one-dimension has several applications related to boundary value problems in ordinary differential equations. In this section the Green's function is introduced in the context of a simple one-dimensional problem.

Consider the differential equation in the standard form (Jeffrey, 2001)

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x) \tag{1.1}$$

which is defined over the interval $a \leq x \leq b$.

Now let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation, with $y_1(x)$ such that at $x = a$ it satisfies the homogeneous boundary condition

$$k_1 y_1(a) + k_2 \acute{y}_1(a) = 0, \tag{1.2}$$

and $y_2(x)$ such that at $x = b$ it satisfies the homogeneous boundary condition

$$k_1 y_2(b) + k_2 \acute{y}_2(b) = 0. \tag{1.3}$$

The solution of equation of (1.1) can be written as

$$y(x) = \int_a^x \frac{y_1(t)y_2(x)}{W(t)} f(t)dt + \int_x^b \frac{y_2(x)y_1(t)}{W(t)} f(t)dt, \tag{1.4}$$

where

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

The solution (1.4) can be written as

$$y(x) = \int_a^b G(x,t) f(t) dt, \tag{1.5}$$

where the function $G(x,t)$ is called the Green's function for differential equation (1.1) described over the interval $a \le x \le b$. This function is defined as

$$G(x,t) = \begin{cases} \dfrac{y_1(t) y_2(x)}{W(t)}, & a \le t \le x \\ \dfrac{y_2(x) y_1(t)}{W(t)}, & x \le t \le b. \end{cases} \tag{1.6}$$

The Green's function in (1.5) has the following properties (Jeffrey, 2001):

1. The piecewise defined Green's function $G(x,t)$ satisfies the differential equation in the respective intervals $a \le x \le t$ and $t \le x \le b$.
2. $G(x,t)$ is continuous function of x for $a \le x \le b$.
3. $G(x,t)$ satisfies the homogeneous boundary conditions.
4. The function $G_x(x,t)$ is continuous for $a \le x \le t$ and $t \le x \le b$, but it is discontinuous across where it experiences the jump

$$G_x(x, x_+) - G_x(x, x_-) = -1 ,$$

where G_x is derivative of G with respect to x.

Green's function in two-dimensions

In this section, we illustrate the use of Green's function in two-dimensions to the boundary value problems in partial differential equations which arise in a wide class of problems in engineering and mathematical physics.

The concept of Green's functions is intimately tied to the Dirac delta function. The Dirac delta function $\delta(x - \xi, y - \eta)$ in two-dimensions is presented by (Rahman, 2007):

$$I.\ \delta(x-\xi,y-\eta)=\begin{cases}\infty, & x=\xi, y=\eta\\ 0 & \text{Otherwise}\end{cases}$$

$$II.\iint_{\Omega_\varepsilon}\delta(x-\xi,y-\eta)\,dxdy=1,\Gamma_\varepsilon:(x-\xi)^2+(y-\eta)^2<\varepsilon^2$$

$$III.\iint_{\Omega} f(x,y)\delta(x-\xi,y-\eta)dxdy=f(\xi,\eta)$$

for arbitrary continuous function $f(x,y)$ in the region Ω.

The application of Green's function in two-dimension can best be described by considering the solution of the Dirichlet problem

$$\nabla^2 u=h(x,y)=0 \qquad \text{in two-dimensional region } \Omega$$

$$u=f(x,y) \qquad \text{on the boundary } \Gamma,$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

The Green's function denoted by $G(x,y;\xi,\eta)$ for the Dirichlet problem involving the Laplace operator is defined as the function which satisfies the following properties:

i- $\Delta G=\delta(x-\xi,y-\eta)$ in Ω , $G=0$ on Γ

ii- G is symmetric, that is, $G(x,y;\xi,\eta)=G(\xi,\eta;x,y)$

iii- G is continuous in $(x,y;\xi,\eta)$ but $\frac{\partial G}{\partial n}$ the normal derivative has a discontinuity at the point (ξ,η) which is specified by the equation

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\frac{\partial G}{\partial n}ds=1,$$

where n is the outward normal to the circle

$$\Gamma_\varepsilon: \qquad (x-\xi)^2+(y-\eta)^2=\varepsilon^2.$$

Theorem 1.1 (Rahman, 2007)

The solution of the Dirichlet problem $\nabla^2 u = h(x,y)$ in Ω with the boundary condition $u = f(x,y)$ on Γ is given by

$$u(x,y) = \iint_{\Omega} G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_{\Gamma} f\frac{\partial G}{\partial n}ds\,, \tag{1.7}$$

where G is the Green's function and n denotes the outward normal to the boundary Γ of the region Ω.

1.2 Background of the problem

In 1828 George Green (1793–1841) published an Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism. Green's essay remained relatively unknown until it was published between 1850 and 1854. In 1877 Carl Neumann considered the concept of Green's functions in his study of the Laplace's equation.

With the function's success in solving Laplace's equation, other equations began also to be solved using Green's functions. In the case of the heat equation, in 1888 Hobson derived the free-space Green's function for one, two and three dimensions. Indeed, Sommerfeld would be the great champion of Green's functions at the turn of the 20th century because he presented the modern theory of Green's function as it applies to the heat equation (Duffy, 2001).

As mentioned, Green's functions have become a fundamental mathematical technique for solving boundary value problems and other important equation in applied mathematics. Properties of Green's functions for bounded region have been investigated in detail by many authors.

We next give a Green's function on unbounded simply connected region.

Consider the Green's function $G(x,y;\xi,\eta)$ that satisfies

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -\delta(\chi - \xi, y - \eta), \quad (x,y) \in \Omega \tag{1.8}$$

with boundary condition as

$$u(x, y) = 0, \quad (x, y) \in \Gamma. \tag{1.9}$$

Here Ω can be bounded or unbounded region and $\delta(x, y)$ is the Diract delta-function. If Ω is the unbounded half-plane $\{-\infty < x < \infty, y > 0\}$, then the classical Green's function is given by (Hon et al., 2010)

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \sqrt{\frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2 + (y-\eta)^2}}. \tag{1.10}$$

Green's function for the disk $|w| < 1$ is given by

$$G(w, s) = -\frac{1}{2\pi} \ln \left| \frac{w - s}{1 - \bar{s}w} \right|, \tag{1.11}$$

where s is inside the unit disk and is a pole of G.

For arbitrary bounded simply connected regions Ω, Green's function can now be found by the method of conformal transplantation. Let $w = f(z)$ map Ω conformally onto $|w| < 1$ and $\Omega \cup \partial\Omega$ continuously onto $|w| \leq 1$. If G is the Green's function for Ω, the function

$$(w, s) \rightarrow g(f^{[-1]}(w), f^{[-1]}(s))$$

has all properties of Green's function for the unit disk, and hence must agree with (1.10). It follows that the desired Green's function for Ω is given by (Henrici, 1986)

$$G(z, z_0) = -\frac{1}{2\pi} \ln \left| \frac{f(z) - f(z_0)}{1 - f(z)f(z_0)} \right|. \tag{1.12}$$

In general the Green's function for Ω can be expressed by

$$G(z, z_0) = u(z) - \frac{1}{2\pi} \ln|z - z_0|, \qquad z, z_0 \in \Omega, \tag{1.13}$$

where u is the unique solution of the interior Dirichlet problem

$$\begin{cases} \nabla^2 u(z) = 0, & z \in \Omega, \\ u(\eta(t)) = \ln|\eta(t) - z_0|, & \eta(t) \in \Gamma. \end{cases} \tag{1.14}$$

Nasser (2007) has developed a new method for solving the interior and exterior Dirichlet problem in simply connected regions with smooth boundaries. His method is based on two uniquely Fredholm integral equations of the second kind with the generalized Neumann kernel. Recently, his method has been used by Alagele (2012) for computing Green's function on bounded simply connected region only. This research wishes to extend the work by Alagele (2012) for computing Green's function on an unbounded simply connected region by getting a unique solution of the exterior Dirichlet problem using integral equation approach with the generalized Neumann kernel.

1.3 Statement of the problem

This research is to compute the Green's function on an unbounded simply connected region by conformal mapping and by solving an exterior Dirichlet problem via an integral equation with the generalized Neumann kernel.

1.4 Objectives of Study

The objectives of this research are:

i. To investigate the properties of Green’s function for unbounded simply connected region and its connection with the exterior Dirichlet problem and conformal mapping.
ii. To compute the Green’s function on an unbounded simply connected region by using conformal mapping method.
iii. To compute Green’s function on an unbounded simply connected region by solving an exterior Dirichlet problem via an integral equation with the generalized Neumann kernel.
iv. To create a numerical technique for solving the boundary integral equation using MATHEMATICA.
v. To compute and graph Green’s functions on several test regions.

1.5 Scope of Study

There are Dirichlet problems and Green’s function for bounded and unbounded multiply connected regions. The main concern of this research is the evaluation of Green’s function on the unbounded simply connected region with smooth boundary. The boundary integral equation method which involves the generalized Neumann kernel and conformal mapping method are considered for a computing Green’s function.

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