

THE APPLICATION OF FINITE ELEMENT METHOD IN BURGERS' EQUATION

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My sincere appreciation also extends to all my colleagues and other who have provided assistance at various occasions. Their views and opinions are helpful indeed. Unfortunately it is impossible to list all of them in this limited space.

Last but not least, I would like to thank my family members who have given me their undying support.

To my beloved father and mother

Adnan bin Kechil

&

Ramlah bte Ngah

And also

To my dearest sister and brothers

Nur Hidayah Adnan

Muhamad Solihin Adnan

Zulkarnain Adnan

Abdul Azim Adnan

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ABSTRACT

The main objective of this project is to provide the explanation of the application of Finite Element method in 1- dimensional Burgers' equation. Burgers' equation captures very important phenomena surrounding world. Its can be categorized as a parabolic, elliptic and hyperbolic partial differential equation. For this reason, Burgers equation is often used as a case study for testing and comparing computational technique. The 1- dimension Burgers' equation is investigated and their results are compared with analytical solution of the problem. To solve the problem in finite element method, few concept are introduced such that calculus of variation, weight function, and weak formulation. A brief explanation of the calculation using Finite Element Method (FEM) is given. In this research, the Finite Element method algorithm is written in Matlab and Maple programme. Based on the numerical results, we can see from the behaviour of the graph , the solution is approaching some limit as we increased the number of Reynolds. The error between numerical and analytical result becomes larger. We cay say that , FEM is a unique and useful technique and it is being extended to solve many problem in engineering and science area.

ABSTRAK

Objektif utama projek ini adalah untuk memberi penjelasan mengenai penggunaan Kaedah Unsur Terhingga dalam persamaan Burgers ' 1 dimensi. Persamaan Burgers memainkan peranan yang penting berkaitan dengan fenomena alam di sekeliling dunia. Ia boleh dikategorikan sebagai persamaan pembezaan separa parabola, elips dan hiperbola. Disebabkan ini, persamaan Burgers sering digunakan sebagai kes kajian untuk menguji dan membandingkan teknik pengiraan melalui kaedah pengkomputeran. Persamaan Burgers ' 1 dimensi diuji dan keputusan akan dibandingkan dengan penyelesaian secara analitikal. Untuk menyelesaikan masalah dalam kaedah unsur terhingga, beberapa konsep akan di aplikasikan seperti kalkulus ubahan, fungsi pemberat dan formulasi lemah. Satu penerangan ringkas mengenai pengiraan menggunakan Kaedah Unsur Terhingga (FEM) diberikan. Dalam kajian ini, Unsur Terhingga Kaedah algoritma ditulis dalam program Matlab dan Maple. Berdasarkan keputusan kaedah unsur terhingga, kita boleh lihat dari sudut perubahan graf, keputusan itu menunjukkan bahawa penyelesaian itu sedang menghampiri sesuatu had apabila bilangan Reynolds di tambah dan perbezaan keputusan diantara kaedah berangka dan analitikal semakin bertambah besar. Secara keseluruhannya, Kaedah Unsur Terhingga (FEM) merupakan teknik yang unik dan sangat berguna. Ia boleh menyelesaikan banyak masalah dalam bidang kejuruteraan dan bidang sains.

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LIST OF SYMBOLS

δ	-Delta
ε	$-\left(\frac{1}{\text{Re}}\right)$ (Constant measuring viscosity)
π	-Pi
$\phi(x)$	-Initial function
$w_o(x)$	-Initial condition
$u(x,t)$	-Velocity
$u_1(t), u_2(t)$	-Boundary condition
t	-Time
x_i	-Nodes
i	-Number of column
j	-Number of row
A,B,M,K	-Matrix
N	-Number of element
$h_i(x)$	-Linear basis function
$h(t)$	-Differentiable function

LIST OF ABBREVIATIONS

FD	-Finite Difference method
FEM	-Finite Element method
RE	-Reynolds number
PDEs	-Partial Differential Equations
ODE45	-Ordinary Differential Equation solver in MATLAB
MATLAB	-Matrix Laboratory (Numerical computing)
MAPLE	-Computer Algebra system

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

The finite element method is a numerical solution technique. It's used to solve complex problems. This method becomes popular over the last decade. The scientists and engineers have used the finite element method (FEM) for the modeling of the complex problem. This method having applicability in many areas of engineering, physical problem (stiffness, density and more) and physics such heat transfer (conduction, convection and radiation), fluid flow, electrical potential problem and many more. Finite element is a mathematical technique for obtaining approximate numerical solutions to the abstract equations of calculus that predicts the response of physical systems subjected to external influence.

The aim of this study is to provide the introduction and implementation of Finite Element Method with Burgers equation in boundary value problems. The application of this method will be introduced by showing the example for one dimensional problem.

The Burger's equation serves as a model for many interesting problem in applied mathematics. The equation is useful for modeling such as shock flow, traffic flow and many more. In fact it can be used to model for any non-linear wave propagation problem subject to dissipation. Depending on the problem, the dissipation can get from viscosity, heat conduction, mass diffusion and others source.

Because of this reason, Burgers' equation is always used as a case study for comparing computational technique. Finally, the comparison can be made in order to validate the expected result as well as its accuracy.

1.2 Statement of the Problem

Nowadays, many researchers in engineering and science already used the finite element method (FEM) and finite difference method (FDM) to solve many problems in many area of engineering and science such as heat transfer, shock flow, traffic flow and so on. Now the problem is how to improve the accuracy between these two methods in solving real world problem.

Finite difference method (FDM) is the simplest method for the solution of boundary value problems. It's used to handle rectangular shape and simple alteration. However it suffers the problem of low accuracy solutions. Finite Element Method (FEM) was introduced which will produce better accuracy results.

However, for the problem of boundary value problems in one-dimension, two-dimension and three dimension when the geometry of the problem are complex deformation of structure and with boundary relative cases, FDM will fail to produce reliable solutions. However, these problems can be eliminated by introducing the Finite element method (FEM) to solve the 1- dimensional Burgers' equation.

1.3 Objective of the Study

The main objectives of this study are:

- i. To introduce the finite element method (FEM), numerical technique for the solution of Burgers equation.
- ii. To solve the 1-dimensional Burgers equation by using finite element method (FEM).
- iii. To compare the accuracy of solution between the numerical solution with analytical solution.

1.4 Scope of the Study

This study is a numerical study of basic theory of the finite element method (FEM). Then, we apply the finite element method (FEM) with Burgers equation. We will investigate the finite element method with Burgers' equation boundary condition in order to see the behavior of the model.

Besides that, this study will discuss about several concepts in finite element method and formulation the Weak Form from the Burgers equation. Basically, the (FEM) can be used to solve boundary value problems in one-dimension, two-dimension and three-dimension. Furthermore, the availability of fast and inexpensive computer allows problems which are intractable using analytic method can be solved numerically. However, in this report we will confine our scope of the boundary value problems in 1 –dimension Burgers' equation.

1.5 Significance of the Study

The result of this study will give benefits to mathematics and engineering fields. In mathematics, the research will widen the application of finite element method in solving the boundary value problems especially in one-dimensional problem.

Besides that, we can see the comparison between the Finite Element Method (FEM) and the analytical value by looking the accuracy of the result. With this result, maybe the researcher can be used as one of references.

The advancement in computer technology enables us to formulate and solve complex system as well as the visualization of solutions quickly and accurately.

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