ROOT COUNTING IN PRODUCT HOMOTOPY METHOD

HAZRINA BINTI TAJUDIN

UNIVERSITI TEKNOLOGI MALAYSIA

ROOT COUNTING IN PRODUCT HOMOTOPY METHOD

HAZRINA BINTI TAJUDIN

A thesis submitted in fulfillment of the
requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JULY 2013

To my beloved family and friends.

Thanks for all the efforts, guidance, tender support and blessings that shower on me.

ACKNOWLEDGEMENT

Firstly, I would like to express my deepest gratitude to Almighty God; because of His Blessings, I am able to complete this thesis successfully and within the stipulated time. I would like to express my appreciation and thanks to my project supervisor, PM Dr. Ali Abd. Rahman for his guidance and constructive criticisms. He has been the determining factor in starting and finishing this research.

Also not forgetting to convey my deeply appreciation to Hassan M. S. Bawazir. The supervision and support that he gave are really appreciated. Without his efforts and guides, I will not be able to understand this research. I learned a lot from this study.

Finally, an honorable mention goes to my family and friends for their understandings and supports in completing my project. Without helps from them, I would face many difficulties while doing my project.

ABSTRACT

Product Homotopy method is used to solve dense multivariate polynomial systems for finding all isolated solutions (real or complex). There are two stages in the computation of Homotopy method which are root counting and root finding. This study focuses on root counting which involves the computation of multi-homogeneous Bézout number (MHBN). This value determines the number of solution path in the second stage. Homogenization of partition each gives its own MHBN. Therefore, it is crucial to have minimum MHBN. The computation of minimum MHBN using local search method, fission and assembly method and genetic algorithm had become intractable when the system size gets larger. Hence, this study applied recent heuristic method, Tabu Search. Other than that, the computation of estimating MHBN is of exponential time. For large size system, the usage of row expansion with memory becomes impossible, hence, this study focus on implementing General Random Path algorithm (GRPA). This study implements Tabu search method and GRPA into several systems of different sizes. Tabu search is effective since the global minimum is obtained instead of the local minimum. Other than that, the number of visited partition is much smaller compared with the previous method. Although GRPA gives estimated value, it helps for large size system. We implement two accuracy level in the computation and in the result, the N=1000 gives more accurate result. Hence, GRPA is important when it comes to solve estimated MHBN for large size system.

ABSTRAK

Kaedah homotopi produk digunakan untuk menyelesaikan sistem padat polinomial berbilang pembolehubah untuk mencari semua penyelesaian terpencil (sebenar atau kompleks). Terdapat dua peringkat dalam pengiraan kaedah homotopi iaitu pengiraan akar dan penemuan akar. Kajian ini memberi tumpuan kepada pengiraan akar yang melibatkan pengiraan nombor Bézout berbilang-homogen (MHBN). Nilai ini menentukan bilangan jalan penyelesaian dalam peringkat kedua. Setiap pengkelasan memberikan MHBN yang berbeza. Oleh itu, penting untuk mempunyai MHBN minimum. Pengiraan MHBN minimum dengan menggunakan kaedah carian tempatan, kaedah pemecahan dan penggabungan serta algoritma genetik tidak boleh digunakan bagi menyelesaikan sistem bersaiz besar. Oleh itu, kajian ini menggunakan kaedah heuristik yang terbaru iaitu 'Tabu Search'. Selain daripada itu,pengiraan MHBN mempunyai pola meningkat secara eksponen dimana nilai MHBN sukar ditentukan apabila berdepan dengan sistem bersaiz besar Oleh itu, penggunaan 'Row Expansion with Memory' menjadi mustahil. Fokus kajian ini adalah menggunakan algoritma 'General Random Path' (GRPA). Kajian ini melaksanakan kaedah Tabu search dan GRPA terhadap beberapa sistem yang berlainan saiz. Tabu search berkesan kerana minimum global diperolehi bukannya minimum tempatan. Selain daripada itu, bilangan pengkelasan yang terlibat adalah jauh lebih kecil berbanding dengan kaedah sebelumnya. Walaupun GRPA cuma memberikan nilai anggaran, ia membantu untuk sistem saiz besar. Kami melaksanakan dua tahap ketepatan dalam pengiraan GRPA dan keputusan N = 1000 memberikan hasil yang lebih tepat. Oleh itu, penggunanan GRPA adalah sangat penting untuk sistem yang bersaiz besar.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

There are many methods to find isolated solutions of multivariate polynomial system. One of the methods is Homotopy method. In general, Homotopy method can be defines as follows:

$$H(z,t) = a(1-t)^k Q(z) + t^k F(z) \tag{1.1}$$

with $\alpha \in C$, $0 \le t \le 1$, $k \in \mathrm{N}$, *Q(z)* represent the start system and the *F(z)* represent the original system. In order to find an isolated solution, the start system must have a known solution.

Homotopy method can be divided into two classes based on the degree of complexity. These are product Homotopy method and polyhedral Homotopy method. In comparison of these two classes, the second class is much complicated. Product Homotopy method consists of two main stages which are root counting and root finding.

Since polynomial systems have different density degree; sparse and dense, product Homotopy method is used to solve for dense polynomial system while polyhedral Homotopy method used for solving a sparse polynomial system. A polynomial system is dense if the density exceeds 35% and sparse otherwise. If the density of the monomial is at 100%, both methods are applicable. Throughout this thesis, the polynomial system considered is as follows:

$$F(z) = (f_1(z),..., f_n(z))^T \tag{1.0}$$

There is a deficient case of the polynomial system. It happens when the number of solution is less than total degree. Deficient case leads the solution system at infinity. Usually, the deficient case happens to the sparse system (Verschelde, 1996).

This study focus on root counting process in product Homotopy method. There are two important things to consider throughout the Homotopy process. These two are efficiency and effectiveness. The Homotopy method is said to be efficient depending on bound on the number of the isolated solution. The bound is the total degree of the polynomial system. The total degree is basically known as the classical Bézout number.

In order to minimize the solution path, the Bézout number will be homogenized. This Bézout number is then called Multi-Homogeneous Bézout number, in short MHBN. The generalization of the total degree (MHBN) will definitely provide sharper bound and hence increase the efficiency of the method. Throughout the entire Homotopy process, it is crucial to obtain the minimum MHBN since the MHBN will determine the number of solution path during the root finding process.

Study by Hassan (2012) mentioned that as system size getting larger, the homogenizing of the system tends to increase in exponential order. This happens since computation of estimated MHBN is equivalent to computing permanent of a matrix. Hence, the ability of finding the MHBN is said to be NP-complete. Previous research had implied methods that are not too practical in nature. Therefore, there is a need to apply the random path algorithm. This study will implement General Random path algorithm into several systems.

Homogenization of MHBN each will gives its own value. The optimal MHBN is the one with the minimum value. Previous methods to find the minimum MHBN are local search method, fission and assembly method and genetic algorithm. These methods become intractable when size of the system gets larger. Hence, the implementation of heuristic method is necessary. Recent method introduced by Hassan (2011) for finding the minimum MHBN is Tabu Search method (TSM). This study will focus on implementing the Tabu Search method.

1.2 Statement of the Problem

In order to find the minimum MHBN, there are many methods that have been tested by previous researchers. The homogenization process of the partition each provides different MHBN. Minimum MHBN will reduce path taken in the root finding process and therefore reduce the cost to run the program. Tabu Search method (TSM) and general random path algorithm (GRPA) is a new method introduced by previous researcher (Hassan, 2011). Among all methods, TSM is the only method that can deal with large system size, n and it also gives the most minimum MHBN.

Second problem involve the computation of estimating MHBN is of exponential time. It is equivalent to compute permanent of a matrix. The usage of random algorithm is necessary for this problem. The present work seeks to implement all of these ideas in the computation of MHBN. We will also test for the effectiveness of these two methods.

1.3 Objectives of the Study

The main objectives of this research are:

i) To implement General Random Path algorithm (GRPA) to compute the estimation of multi-homogeneous Bézout number (MHBN).

ii) To implement Tabu Search method (TSM) for the computation of minimum MHBN.

iii) To apply these two methods into several systems with different sizes.

iv) To test for effectiveness of the methods based on a particular example.

1.4 Scope of the Study

There are two types of Homotopy method which are Product Homotopy method and Polyhedral Homotopy method. This study only takes into consideration the Product Homotopy method which focuses on solving the dense type polynomial system.

The study focus on the first stage of Homotopy method which is the root counting. Firstly, we will focus on estimating MHBN for large size system using GRPA. Then, the study continues with the computation of the minimum MHBN using TSM. MHBN is known as the generalization of the total degree (TD). Compared to TD, MHBN is said to give sharper bound on the number of solution path. This study will imply the usage of Tabu Search method (TSM) to find the minimum MHBN.

In order to compute the MHBN, several systems will be use throughout this study. 7 systems were chosen out of 20 systems from previous researcher (Hassan, 2011). Other than that, the effectiveness of the method is also tested on particular example.

1.5 Significance of the Study

In order to solve polynomial system using Homotopy method, there are two stages that have to be taken into consideration which are the root counting and root finding. Root counting involves the search of estimating the minimum Multi Homogeneous Bézout Number (MHBN) which is to be used in the second stage. After MHBN is calculated, the start system can be constructed. The study will implement two methods which are GRPA, which is use to estimate MHBN of large size system, and TSM which is use to find minimum MHBN. These two methods will be applied into several systems listed in the next chapter. We will also analyze the effectiveness of the method.

1.6 Dissertation Outline

This study focus on the root counting process in product Homotopy method. Recent method had become intractable to solve systems of large size. In order to compute for minimum path which determined by MHBN, recent heuristic method, Tabu search is applied. Other than that, General Random Path algorithm is also implemented in this study to solve MHBN for large size system. This method searches MHBN of a specific partition. All MHBN can be used in the root finding process. The advantage of using minimum MHBN is that it can save computation time.

Chapter 2 will focus on previous research on product Homotopy method. There are some method used previously in root counting process such as local search method, fission and assembly method and genetic algorithm.

Chapter 3 will discuss on the method used in this study. The methods are Tabu Search and General Random Path algorithm. This chapter will present in detail the concept these two methods. Several systems that will be used in the computation are also listed in this chapter.

Chapter 4 will provide results of the implementation of these two methods. This chapter will also present the comparison of previous method with Tabu Search to check for the effectiveness of Tabu Search.

Chapter 5 will gives the summary of results obtained in this study. This chapter will also provide recommendation for future research.

REFERENCES

Garcia, C. B. and Zangwill, W. L.. Finding all solutions to polynomial systems and other systems of equation. *Mathematics Programming.* 1979. 16: 159-176.

Hassan Mohamed S. B. *Homotopy method for finding all isolated solutions of multivariate polynomial systems*. Ph.D. Thesis. Universiti Teknologi Malaysia. 2011.

Hassan Mohamed S. B., Ali A. R. and Nor'Aini A.. A Generalized Permanent Estimator and its Application in Computing Multi-Homogeneous Bézout Number. *Research Journal of Applied Sciences, Engineering and Technology.* 2012. 4(23):5206-5211.

Hassan Mohamed S. B. and Ali A. R.. A Tabu Search Method for Finding Minimal Multi-Homogeneous Bézout Number. *Journal of Mathematics and Statistics.* 2010. 6(2):105-109.

Kotsireas, I. S.. Homotopies and Polynomial System Solving I: Basic Principles. *ACM SIGSAM Bulletin.* 35(1).

Li, T. Y. and Bai, F.. Minimizing Multi Homogeneous Bézout Number by a Local Search Method. *Mathematics Computational.* 2000. 70: 767-787.

Li, T. Y., Sauer, T. and Yorke, J. A.. The Random Product Homotopy Method and Deficient Polynomial Systems. *Numerical Mathematics.* 1987. 51:481-500.

Li, T. Y., Sauer, T. and Yorke, J. A.. Numerical Solution of a Class of Deficient Polynomial System. *SIAM J. Numerical Analysis.* 1987. 24(2): 435-451.

Li, T. Y. and Wang, X.. Solving Deficient Polynomial Systems with Homotopies which keep the subschemes at infinity invariant. *Mathematics of Computation.* 1991. 56(194):693-710.

Liang, H. S. Linsong, B. and Xiaoyan, L.. Random path method with pivoting for computing permanent of a matrices. *Applied Mathematics Computation.* 2007. 185:59-71.

Malajovich, G. and Meer, K.. Computing Multi-Homogeneous Bézout Number is hard. *Theory Computational System.* 2007. 553-570.

Morgan, A. and Sommesse, A.. A Homotopy for Solving General Polynomial Systems That Respects m-Homogeneous Structures. *Applied Mathematics and Computation.* 1987. 24:101-113.

Norman Biggs. *Discrete Mathematics.* 2nd edition. United States: Oxford University Press Inc., New York. 2002.

Verscehlde, J. *Homotopy continuation method for solving polynomial systems*. Ph. D. Thesis. Katholieke Universiteit Leuven. 1996.

Verscehlde, J. and Haegemans, A. Homotopies for solving polynomial systems within a bounded domain. *Theoretical Computer Sciences.* 1994. 133, 165-185.

Wampler, C. W. Bézout number calculations for multi-homogeneous polynomial systems. *Applied Mathematics and Computation.* 1992. 51, 143-157.