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Computing the Exterior Center of Metacyclic *p*-groups of Nilpotency Class at Least Three

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Abstract: A group G is metacyclic if it contains a cyclic normal subgroup K such that G/K is also cyclic. Metacyclic p-groups classified by different authors. A group is called capable if it is a central factor group. The purpose of this study is to compute the exterior center of finite non-abelian metacyclic *p*-groups, *p* is an odd prime, for some small order groups using Groups, Algorithms and Programming (GAP) software. We also determine which of these groups are capable.

Key words: *p*-groups, metacyclic groups, exterior center, capable groups

INTRODUCTION

A group G is metacyclic if there is a cyclic normal subgroup K whose quotient group G/K is also cyclic. The presentation of finite metacyclic groups contains two generators and three defining relations. Much attention have been given to some specific types of metacyclic groups. Metacyclic groups with cyclic commutator quotient discussed by Zassenhaus and Hall. Metacyclic *p*-groups of odd order have been classified by Beyl, King, Liedahl, Newman and Xu, Rdei and Lindenberg. The result that every finite metacyclic group can be decomposed naturally as a semidirect product of two Hall subgroups made by Sim (1994) was an important progress (Hempel, 2000).

Beuerle (2005) classified all finite non-abelian metacyclic *p*-groups. The main objective of this study is to compute the exterior center of finite non-abelian metacyclic *p*-groups, *p* is an odd prime, for some small order groups by using Groups, Algorithms and Programming (GAP) software, based on Beuerle's classification.

Before we introduce the exterior center for a group, we need to introduce the non-abelian tensor square as follows:

Definition 1 (Brown *et al.*, 1987): For a group G, the non-abelian tensor square, $G\otimes G$, is generated by the symbols $g\otimes h$ where $g, h\in G$ subject to the relations:

$$gg'\otimes h = ({}^{g}g'\otimes {}^{g}h)\,(g\otimes h) \text{ and}$$
$$g\otimes hh' = (g\otimes h)\,({}^{h}g\otimes {}^{h}h')$$

for all $g, g', h, h'\in G$, where ${}^{h}g = hgh^{-1}$ denotes the conjugate of g by h.

The exterior square is a factor group of the tensor square, defined as follows:

Definition 2 (Brown *et al.*, 1987): For any group G the exterior square of G is defined as $G\wedge G\ (G\otimes G)/\nabla(G)$, where $\nabla(G) = \langle x\otimes x: x\in G\rangle$ and $\nabla(G)$ is a central subgroup of $G\otimes G$.

The tensor center of a group was defined by Ellis (1995) as:

$$Z^{\otimes}(G) = \{a\in G: a\otimes g = 1_{\otimes}: \forall g\in G\}$$

Similarly, the exterior center of a group is defined as:

$$Z^{\wedge}(G) = \{a\in G: a\wedge g = 1_{\wedge}: \forall g\in G\}$$

Here, $1_{\otimes}$ and $1_{\wedge}$ denote the identities in $G\otimes G$ and $G\wedge G$, respectively. It can be easily shown that $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ are characteristic and central subgroups of G with $Z^{\otimes}(G)\subseteq Z^{\wedge}(G)$.

Baer (1938) initiated a systemic investigation of the question in which conditions a group G must fulfill in order to be the group of inner automorphisms of some group H, i.e., $G\cong H/Z(H)$. Baer was the first to study the notion of the capability of a group. Hall and Senior (1964)

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introduced the term by defining a capable group as equal to its central factor group. The capability for some groups has been studied by many authors including Baer (1938) who characterized finitely generated abelian groups which are capable in the following theorem:

Theorem 1 (Baer, 1938): Let A be a finitely generated abelian group written as $A = \mathbb{Z}_{n_1} \otimes \mathbb{Z}_{n_2} \otimes \cdots \otimes \mathbb{Z}_{n_k}$, such that each integer n_{i+1} is divisible by n_i and $\mathbb{Z}_0 = \mathbb{Z}$ the infinite cyclic group. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

A necessary and sufficient condition for a group to be a capable has been established by Beyl *et al.* (1979) in terms of the epicenter, defined as follows:

Definition 3 (Beyl *et al.*, 1979): The epicenter, $Z^*(G)$ of a group G is defined as:

$$\bigcap\{\phi Z(E); (E, \phi) \text{ is a central extension of } G\}$$

The epicenter can be easily seen as a characteristic subgroup of G contained in its center. The following criterion now characterizes capable groups.

Theorem 2 (Beyl *et al.*, 1979): A group G is capable if and only if $Z^*(G) = 1$.

Beyl *et al.* (1979) used this characterization to describe the capable finite metacyclic groups and determine the extra special p-groups which are capable.

By Beuerle (2005), a metacyclic *p*-group is usually given by a presentation of the form:

$$G = \langle a, b;\ a^{p^m} = 1,\ b^{p^n} = a^k,\ bab^{-1} = a^r \rangle$$

where, m, $n \geq 0$, $r > 0$, $k \leq p^m$, $p^m | k(r-1)$ and $p^m \mid r^{p^n} - 1$.

Metacyclic *p*-groups are divided by Beuerle (2005) into two main classes as follows:

$$G_p(\alpha, \beta, \varepsilon, \delta, \pm) = \langle a, b;\ a^{p^\alpha} = 1,\ b^{p^\beta} = a^{p^{\alpha-\varepsilon}},\ bab^{-1} = a^r \rangle$$

where, $r = p^{\alpha-\delta}+1$ or $r = p^{\alpha-\delta}-1$. We say that the group is of positive or negative type if $r = p^{\alpha-\delta}+1$ or $r = p^{\alpha-\delta}-1$, respectively.

For short, we write G_+ and G_- for $G_p(\alpha, \beta, \varepsilon, \delta, +)$ and $G_p(\alpha, \beta, \varepsilon, \delta, -)$, respectively.

The following definition gives a standardized parametric presentation of metacyclic *p*-groups.

Definition 4 (Beuerle, 2005): Let Gbe a finite metacyclic *p*-group. Then there exist integers α, β, δ, ε with α, $\beta > 0$ and δ, ε nonnegative, where $\delta \leq \min\{\alpha-1, \beta\}$ and $\delta+\varepsilon \leq \alpha$ such that for an odd prime *p*, $G_p(\alpha, \beta, \varepsilon, \delta, +)$. If p = 2, then in addition $\alpha-\delta>1$ and $G_2(\alpha, \beta, \varepsilon, \delta, +)$ or $G_p(\alpha, \beta, \varepsilon, \delta, -)$, where in the second case $\varepsilon \leq 1$. Moreover, if G is of positive type, then $\delta > 0$ for all *p*.

Some of the basic properties and various subgroups are given in the next proposition:

Proposition 1 (Beuerle, 2005): Let G be a group of type $G_p(\alpha, \beta, \varepsilon, \delta, \pm)$ and $k \geq 1$. Then we have the following results for the order of G, the center of G, the order of the center of G, the derived subgroups of G, the order of the derived subgroups of G and the exponent of G:

	G_+	G_-
$\lvert G_\pm \rvert$	$p^{\alpha+\beta}$	$p^{\alpha+\beta}$
$Z(G_\pm)$	$\langle a^{p^\delta}, b^{p^\delta} \rangle$	$\langle a^{2^{\alpha-1}}, b^{2^{\max\{1,\delta\}}} \rangle$
$\lvert Z(G_\pm) \rvert$	$p^{\alpha+\beta-2\delta}$	$2^{1+\beta-\max\{1,\delta\}}$
$\gamma_{k+1}(G_\pm)$	$\langle a^{p^{k(\alpha-\delta)}} \rangle$	$\langle a^{2^k} \rangle$
$\lvert \gamma_{k+1}(G_\pm) \rvert$	$p^{\max\{0, k\delta-(k-1)\alpha\}}$	$2^{\max\{0, \alpha-k\}}$
$\mathrm{Exp}(G_\pm)$	$p^{\max\{\alpha, \beta+\varepsilon\}}$	$p^{\max\{\alpha, \beta+\varepsilon\}}$

Moreover, if $\beta > \max\{1, \delta\}$ then $Z(G_-)$ is cyclic if and only if $\varepsilon = 1$.

Though the criterion for capability stated in theorem 2 is easily formulated, the implementation is another matter. As in all cases before, this still requires the cumbersome process of evaluating the factor groups. The connection between the epicenter and the exterior center given by Ellis (1998) provides a much more efficient procedure to determine capability once the non-abelian tensor square is known. The desired external characterization of the epicenter is obtained as follows:

Theorem 3 (Ellis, 1998): For any group G, the epicenter coincides with the exterior center, i.e., $Z^*(G) = Z^\wedge(G)$.

Several authors used the tensor squares method to determine the capability. Beuerle and Kappe (2000) characterized infinite metacyclic groups which are capable.

By using the criterion stated in Theorem 2 and the relationship between the epicenter and exterior center stated in theorem 3, we determine which of them are capable.

Groups, Algorithms and Programming (GAP) software is used as a tool to verify the results found in determining the capability of the groups studied. GAP is a system for computational discrete algebra, with emphasis on computational group theory. GAP provides a programming language, a library of functions that implement algebraic algorithms written in the GAP language as well as libraries of algebraic objects such as for all non-isomorphic groups up to order 2000 (Rainbolt and Gallian, 2010).

THE CLASSIFICATION OF METACYCLIC *p*-GROUPS OF NILPOTENCY CLASS AT LEAST THREE

In this section, the classification of all finite non-abelian metacyclic *p*-groups of nilpotency class at least 3 where *p* is an odd prime, done by Beuerle is stated.

Theorem 4 (Beuerle, 2005): Let *p* be an odd prime and G a metacyclic *p*-group of nilpotency class at least three. Then G is isomorphic to exactly one group in the following list:

$$G \cong \langle a,b;\ a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\delta}} \rangle \tag{1}$$

where, $\alpha, \beta, \delta \in \mathbb{N}$, $\delta\text{-}1 \leq \alpha < 2\delta$, $\delta \leq \beta$.

$$G \cong \langle a,b;\ a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\delta}}, [b,a] = a^{p^{\alpha-\delta}} \rangle \tag{2}$$

where $\alpha, \beta, \delta \in \mathbb{N}$, $\delta\text{-}1 \leq \alpha < 2\delta$, $\delta \leq \beta+\varepsilon$.

If p = 3 and $\alpha = \beta = \delta = 1$ satisfying the conditions of type 1, then the group $G \cong \langle a, b;\ a^3 = b^3 = 1, [b,a] = a \rangle$ is not metacyclic.

For type 2, if p = 3, $\alpha = 5$ and $\beta = \delta = \varepsilon = 3$ satisfying the conditions of the type, then the group $G \cong \langle a, b;\ a^{243} = 1, b^{27} = a^9, [b, a] = a^9 \rangle$ is not metacyclic. Therefore, not all groups of type 1 and 2 are metacyclic groups.

Now, we just need to consider the conditions of metacyclic *p*-groups stated in definition 4 in order to come up with restrictions in the parameters so that nonmetacyclic groups of type 1 and 2 reduced. So, using definition 4, theorem 4 can be straightforwardly rewritten as follows:

Theorem 5: Let *p* be an odd prime and G a metacyclic *p*-group of nilpotency class at least three. Then G is isomorphic to exactly one group in the following list:

$$G \cong \langle a,b;\ a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\delta}} \rangle \tag{3}$$

where, $\alpha, \beta, \delta \in \mathbb{N}$, $\delta \leq \alpha < 2\delta$, $\delta \leq \beta$, $\delta \leq \beta$, $\delta \leq \min\{\alpha\text{-}1, \beta\}$

$$G \cong \langle a,b;\ a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\delta}}, [b,a] = a^{p^{\alpha-\delta}} \rangle \tag{4}$$

where, $\alpha, \beta, \delta \in \mathbb{N}$, $\delta+\varepsilon \leq \alpha < 2\delta$, $\delta \leq \beta$, $\alpha < \beta+\varepsilon$, $\delta \leq \min\{\alpha\text{-}1, \beta\}$.

EXTERIOR CENTER COMPUTATIONS

Here, we use GAP to develop appropriate coding for computing the exterior center for some small order groups of type 3 and 4. We provide GAP programmes to generate general codes and examples of groups of type 3 and 4.

Type 3: First we use GAP programme to generate all finite non-abelian metacyclic *p*-groups of type 3 and construct some examples.

Generating type 3: GAP coding to generate all finite non-abelian metacyclic *p*-groups of type 3 is developed as follows:

```
Type 3: =
function (p, l ,beta ,n)
local F, m, a, b, G;
for m in [1..beta] do
    if n<=l and l<2*n and m>=n and n<=Minimum(l-1, m) and n>=1
    then
    F: = Free group (2);
    a: = F.1; b: = F.2
    G: = F/[a^(p^l), b^(p^m), (b*a*b^(-1))^(-1)*a^(p^(l-n)+1)];
    Print("p=", p," alpha=", l, " beta = ", m, " delta = ", n, " Order (G):",
    Order(G)," Order of Exteriorcenter(G):", Order (Exterior
    Center(G)),"\n");
    fi;
od;
end;
```

We define the group with four parameters which are p, α, β and δ. The GAP code below is used to create the presentation of the group.

```
F: = Free group (2)
a: = F.1;b:=F.2
G: = F/[a^(p^l), b^(p^m), (b*a*b^(-1))^(-1)*a^(p^(l-n)+1)]
```

Constructing examples of type 3: Now, we define the parameters and put them in GAP. For example, we define $p = \alpha = 3$, $\beta = 2, 3, 4$ and $\delta = 2$. First we need to read the file and call the function.

```
gap>Read ("metacyclic_type 3.g")
gap>type 3 (3, 3, 4, 2);
p = 3 alpha = 3 beta = 2 delta = 2 order (G): 243 order of exteriorcenter (G): 3
p = 3 alpha = 3 beta = 3 delta = 2 order (G): 729 order of exteriorcenter (G): 1
p = 3 alpha = 3 beta = 4 delta = 2 order (G): 2187 order of exteriorcenter (G): 3
```

Using the coding as before, we got the following results.

```
gap>type 3 (3, 4, 4, 3);
p = 3 alpha = 4 beta = 3 delta = 3 order (G): 2187 order of exteriorcenter (G): 3
p = 3 alpha = 4 beta = 4 delta = 3 order (G): 6561 order of exteriorcenter (G): 1
gap>type 3 (3, 5, 4, 3);
p = 3 alpha = 5 beta = 3 delta = 3 order (G): 6561 order of exteriorcenter (G): 9
gap>type 3 (5, 3, 4, 2);
p = 5 alpha = 3 beta = 2 delta = 2 order (G): 3125 order of exteriorcenter (G): 5
gap>type 3 (7, 3, 2, 2);
p = 7 alpha = 3 beta = 2 delta = 2 order (G): 16807 order of exteriorcenter (G): 7
gap>Log To();
gap>quit;
```

Summary of results for type 3: For short, we write Ex (G) for the exterior center of G. A summary of GAP results for type 3 is given in Table 1.

Type 4: With similar objective as for groups of type (3), we produce GAP algorithms for groups of type (4).

Generating type 4: We develop the GAP coding to generate all finite non-abelian metacyclic p-groups of type 4.

```
Type 4: =
function (p, l, beta, s, n)
local F, m, a, b, G;
for m in [1..beta] do
if n+s< = l and l<2*n and m> = n and n> = 1 and s> = 1 and l<m+s and
n< = Minimum(l-1, m) then
F: = Free Group (2);
a: = F.1; b: = F.2; G: = F/[a^(p^l), b^(p^m)*a^((-p)^(l-s)), (b*a*b^(-1)*a^(-1))^(-1)*a^(p^(l-n))];
Print ("p = ", p, " alpha = ", l, " beta = ", m, "epsilon = ", s, " delta = ", n, " Order (G):", Order (G), " Order of Exteriorcenter (G):", Order(ExteriorCenter(G)), "\n");
        fi;
    od;
end;
```

We define the group with five parameters which are p, α, β, ε and δ. The GAP code below is used to create the presentation of the group.

```
F: = Free group (2);
a: = F.1; b: = F.2; G: = F/[a^(p^l), b^(p^m)*a^((-p)^(l-s)),
(b*a*b^(-1)*a^(-1))^(-1)*a^(p^(l-n))];
```

Constructing examples of type 4: Now, we define the parameters and put them in GAP. For example, we define p = α = 3, β = 3, 4, 5, 6, ε = 1 and δ = 2. First we need to read the file and call the function.

```
gap> Read ("metacyclic_type4.g");
gap> type 4 (3, 3, 7, 2, 1);
p = 3 alpha = 3 beta = 3 epsilon = 1 delta = 2 Order (G): 729 Order of ExteriorCenter (G): 9
```

Table 1: GAP results for type 3

#	p	α	β	δ	|G|	|Ex (G)|
1	3	3	2	2	243	3
2	3	3	3	2	729	1
3	3	3	4	2	2187	3
4	3	4	3	3	2187	3
5	3	4	4	3	6561	1
6	3	5	3	3	6561	9
7	5	3	2	2	3125	5
8	7	3	2	2	16807	7

Table 2: GAP results for type 4

#	p	α	β	ε	δ	|G|	|Ex (G)|
1	3	3	3	1	2	729	9
2	3	3	4	1	2	2178	27
3	3	3	5	1	2	6561	81
4	3	4	6	1	3	19683	243

```
p = 3 alpha = 3 beta = 4 epsilon = 1 delta = 2 order (G): 2187 order of ExteriorCenter (G): 27
p = 3 alpha = 3 beta = 5 epsilon = 1 delta = 2 order (G): 6561 order of ExteriorCenter (G):81
p = 3 alpha = 3 beta = 6 epsilon = 1 delta = 2 order (G): 19683 order of ExteriorCenter (G): 243
gap> Log To ();
gap> quit;
```

Summary of results for type 4: A summary of GAP results for type (4) is given in Table 2.

CAPABILITY DETERMINATION

Table 1 shows that the groups of type 3 have trivial exterior center with $\alpha = \beta$. Table 2 shows that all groups of type 4 have nontrivial exterior center. Therefore, by Theorem 2 and Theorem 3, we conclude from Table 1 that group 2 and group 4 of type 3 are capable and the rest are not capable. Similarly, from Table 2 we conclude that all groups of type 4 are not capable.

CONCLUSION

In this study we have computed the exterior center of finite non-abelian metacyclic p-groups, p is an odd prime, for some small order groups and determined their capability. We showed that the exterior center of groups of type 3 is trivial with $\alpha = \beta$ and the exterior center of all groups of type 4 is nontrivial. Also, we showed that the groups of type 3 are capable with $\alpha = \beta$ whereas all the groups of type 4 are not capable. It is worth to mention that the order of all groups computed satisfies $p^{\alpha+\beta}$ as stated in proposition 1.

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