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Boundary integral equations with the generalized Neumann kernel for Laplace's equation in multiply connected regions

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ABSTRACT

This paper presents a new boundary integral method for the solution of Laplace's equation on both bounded and unbounded multiply connected regions, with either the Dirichlet boundary condition or the Neumann boundary condition. The method is based on two uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel. Numerical results are presented to illustrate the efficiency of the proposed method.

1. Introduction

In the present paper we shall pursue our investigations [19,20,24,25] of the connections of the elliptic boundary value problems and boundary integral equations with the generalized Neumann kernel. We consider Laplace's equation $\Delta u = 0$ in both bounded and unbounded multiply connected regions G in the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with either the Dirichlet boundary condition or the Neumann boundary condition on the boundary $\Gamma := \partial G$. Under suitable assumptions, both the Dirichlet problem and the Neumann problem have unique solutions (see the standard texts [6,15,16]).

One of the classical methods for solving Laplace's equation is the boundary integral method. For example, a second kind Fredholm integral equation can be derived for the Dirichlet problem by writing its solution as a double layer potential. For bounded simply connected regions, the integral equation is uniquely solvable and its kernel is known as the Neumann kernel (see e.g., [10, p. 280] and [15, p. 130]). However, the integral equation is not uniquely solvable for bounded multiply connected regions. When the connectivity of the region is $m + 1$, the number of linearly independent solution of the homogeneous equation is equal to m. To solve the integral equation, extra constraints on the solution of the integral equation are imposed. An example of such constraints is given by Mikhlin [15, p. 146] (see also [7,9]).

Recently, the interplay of Riemann–Hilbert problems and integral equations with the generalized Neumann kernel has been investigated in [20,24] for simply connected regions with smooth and piecewise smooth boundaries and in [19,25] for bounded and unbounded multiply connected regions. By treating conformal mapping as Riemann–Hilbert problem, integral equations with the generalized Neumann kernel have been implemented successfully in [17,18] for computing the conformal mapping of bounded and unbounded multiply connected regions onto the classical canonical slit domains.

This paper presents two uniquely solvable integral equations with the generalized Neumann kernel to solve the Dirichlet problem and the Neumann problem. We shall prove that the eigenvalues of the kernel of our integral equations are real. This

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is in contrast with the integral equation of Mikhlin's method which has complex eigenvalues (see [7]). In [7,9], the integral equation of Mikhlin's method is discretized by the Nyström method to obtain a linear system. In order to obtain a matrix whose eigenvalues are better distributed for the GMRES iterative method, a preconditioner have been used. For our method, we do not need to use a preconditioner since the eigenvalues of the matrices, obtained by discretizing our integral equations, are real and clustered around 1.

The plan of the paper is as follows: In the next section we present some notations and auxiliary material. In Sections 3 and 4, we derive and study the integral equation with the generalized Neumann kernel and its adjoint kernel respectively. Some theorems related to the eigenvalues of the generalized Neumann kernel are presented in Section 5. In Section 6, we present two methods for calculating the solution the Dirichlet problem. Similar treatment for the Neumann problems is presented in Section 7. Numerical examples will be given in Section 8 and a short conclusion is given in Section 9.

2. Notations and auxiliary material

We consider multiply connected regions G in the extended complex plane $\overline{\mathbb{C}}$ of the following two types:

(a) Bounded region G, of connectivity $m+1 \geqslant 1$, with boundary $\Gamma = \cup_{j=0}^{m}\Gamma_j$ consisting of $m+1$ smooth closed Jordan curves Γ_j, $j = 0,1,2,\ldots,m$. The curve Γ_0 contains the other curves $\Gamma_1,\ldots,\Gamma_m$. The complement $G^- := \overline{\mathbb{C}} \setminus \overline{G}$ consists of m bounded simply connected components G_j interior to Γ_j, $j = 1,2,\ldots,m$, and an unbounded simply connected component G_0 exterior to Γ_0 (see Fig. 1). We assume that α is a fixed point in G.

(b) Unbounded region G, of connectivity $m \geqslant 1$, with boundary $\Gamma = \cup_{j=1}^{m}\Gamma_j$ consisting of m smooth closed Jordan curves Γ_j, $j = 1,2,\ldots,m$. The complement $G^- := \overline{\mathbb{C}} \setminus \overline{G}$ consists of m bounded simply connected components G_j interior to Γ_j, $j = 1,2,\ldots,m$ (see Fig. 2).

The orientation of the boundary $\Gamma = \partial G$ is such that G is always on the left of Γ. Thus, the curves $\Gamma_1,\ldots,\Gamma_m$ always have clockwise orientations. For bounded G, the curve Γ_0 has a counterclockwise orientation. The curve Γ_j is parametrized by a 2π-periodic twice continuously differentiable complex function $\eta_j(t)$ with non-vanishing first derivative

$$\dot{\eta}_j(t) = d\eta_j(t)/dt \neq 0, \quad t \in J_j := [0, 2\pi], \tag{1}$$

$j = 0$ (for bounded G), $1,2,\ldots,m$. The total parameter domain J is the disjoint union of the intervals J_j. We define a parametrization of the whole boundary Γ as the complex function η defined on J by

$$\eta(t) := \begin{cases} \eta_0(t), & t \in J_0 \text{ (for bounded } G), \\ \eta_1(t), & t \in J_1, \\ \vdots & \\ \eta_m(t), & t \in J_m. \end{cases} \tag{2}$$

Let H be the space of all real Hölder continuous functions on the boundary Γ. In view of the smoothness of η, a function $\phi \in H$ can be interpreted via $\hat{\phi}(t) := \phi(\eta(t))$, $t \in J$, as a real Hölder continuous 2π-periodic functions $\hat{\phi}(t)$ of the parameter $t \in J$, i.e.,

$$\hat{\phi}(t) := \begin{cases} \hat{\phi}_0(t), & t \in J_0 \text{ (for bounded } G), \\ \hat{\phi}_1(t), & t \in J_1, \\ \vdots & \\ \hat{\phi}_m(t), & t \in J_m, \end{cases} \tag{3}$$

with real Hölder continuous 2π-periodic functions $\hat{\phi}_j$ defined on J_j; and vice versa.

Here and in what follows, for complex-valued or real-valued functions ψ defined on the boundary Γ and for $t \in J$, we will not distinguish between $\psi(\eta(t))$ and $\psi(t)$. For $t \in J_k$, the values $\psi(t)$ will be denoted by $\psi_k(t)$.

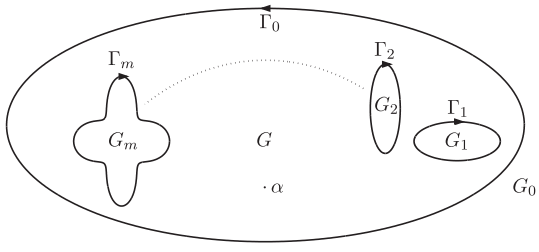

Fig. 1. A bounded multiply connected region G of connectivity $m+1$.

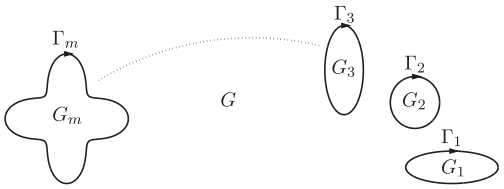

Fig. 2. An unbounded multiply connected region G of connectivity m.

For a given function $\gamma \in H$, the Dirichlet problem and the Neumann problem are defined as follows (see e.g. [3, p. 307], [15, p. 145] and [16, p. 164]):

Dirichlet problem:

Determine a function u harmonic in G, continuous on the closure $\overline{G}$, such that its boundary values satisfy on Γ

$$u(\eta(t)) = \gamma(t), \quad \eta(t) \in \Gamma. \tag{4}$$

For unbounded G, the function u is also required to satisfy $u(z) \to C$ as $|z| \to \infty$ with a constant C.

Neumann problem:

Determine a function u harmonic in G, continuous on the closure $\overline{G}$, such that its boundary values satisfy on Γ

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \quad \eta(t) \in \Gamma, \tag{5}$$

where $\mathbf{n}$ is the exterior normal to Γ and $\gamma \in H$ is a given function such that

$$\int_J \gamma(t)|\dot{\eta}(t)|dt = 0. \tag{6}$$

The function u is also required to satisfy for bounded G the additional condition $u(\alpha) = 0$ and for unbounded G the additional condition $u(z) \to 0$ as $|z| \to \infty$.

The Dirichlet problem and the Neumann problem are uniquely solvable. The unique solution u of the Dirichlet problem or the Neumann problem can be regarded as a real part of an analytic function F in G which is not necessary single-valued. However, the function F can be written as:

$$F(z) = f(z) - \sum_{j=1}^{m} a_j \log(z - z_j), \tag{7}$$

where f is a single-valued analytic function in G, each z_j is a fixed point in G_j, $j = 1,2,\ldots,m$; and $a_1,\ldots,a_m$ are real constants uniquely determined by γ (see [15, p. 149], [16, p. 174] and [21, p. 527].) Without lost of generality, we assume for bounded G that $\mathrm{Im}\, f(\alpha) = 0$ and for unbounded G that $\mathrm{Im}\, f(\infty) = 0$. The constants $a_1,\ldots,a_m$ are chosen to ensure that (see [10, p. 222] and [12, p. 88])

$$\int_{\Gamma_k} f'(\eta)d\eta = 0, \quad k = 1,2,\ldots,m.$$

Since $\Gamma_1,\ldots,\Gamma_m$ are clockwise oriented and

$$F'(z) = f'(z) - \sum_{j=1}^{m} a_j \frac{1}{z - z_j},$$

the Cauchy integral formula implies that the constants $a_1,\ldots,a_m$ are related to the function $F'(z)$ by

$$a_j = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} F'(\eta)d\eta, \quad j = 1,2,\ldots,m. \tag{8}$$

Since $\int_\Gamma f'(\eta)d\eta = 0$ and $\int_\Gamma F'(\eta)d\eta = 0$, thus for unbounded G, the constants $a_1,\ldots,a_m$ satisfy

$$\sum_{j=1}^{m} a_j = \sum_{j=1}^{m} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} F'(\eta)d\eta = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} F'(\eta)d\eta = 0.$$

For bounded G, we define the real constant a_0 by

$$a_0 := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} F(\eta)d\eta = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} F'(\eta)d\eta - \sum_{j=1}^{m} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} F'(\eta)d\eta = -\sum_{j=1}^{m} a_j. \tag{9}$$

In this paper, we shall present the following two methods for calculating the values of the complex function $F(z)$ and hence the values of the real function $u(z) = \mathrm{Re}\, F(z)$:

Method I. The first method is based on using a boundary integral equation with the generalized Neumann kernel. In this method, we calculate first the values of the single-valued analytic function $f(z)$. Then we calculate the values of the multi-valued analytic function $F(z)$ from (7).

Method II. The second method is based on using a boundary integral equation with the adjoint generalized Neumann kernel. In this method, we calculate first the boundary values of the single-valued analytic function F'. Then, we calculate the values of the multi-valued analytic function $F(z)$ as an anti-derivative of F' (see [16, p. 198] and [21, p. 547].) For bounded G, we have

$$F(z) = -\frac{1}{2\pi i}\int_\Gamma F'(\eta)\log\left(1-\frac{z-\alpha}{\eta-\alpha}\right)d\eta + F(\alpha), \tag{10a}$$

where the branch of logarithm is chosen which is equal to zero for $z = \alpha$. For unbounded G, we have

$$F(z) = -\frac{1}{2\pi i}\int_\Gamma F'(\eta)\log\left(1-\frac{\eta}{z}\right)d\eta + F(\infty), \tag{10b}$$

where the branch of logarithm is chosen which is equal to zero for $z = \infty$.

Both methods require determining the values of the real constants $a_1,\ldots,a_m$. These constants are known for the Neumann problem. For the Dirichlet problem, we need to calculate these constants as we shall explain in Theorem 3.

3. The integral equation

Let the function A be defined by

$$A(t) = \Pi(\eta(t)), \tag{11}$$

where Π is the complex-valued function defined for $z \in \mathbb{C}$ by

$$\Pi(z) := \begin{cases} z-\alpha, & \text{if } G \text{ is bounded,} \\ 1, & \text{if } G \text{ is unbounded,} \end{cases} \tag{12}$$

with a fixed point $\alpha \in G$. The generalized Neumann kernel formed with A is defined by

$$N(s,t) := \frac{1}{\pi}\mathrm{Im}\left(\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right). \tag{13}$$

We define also a real kernel M by

$$M(s,t) := \frac{1}{\pi}\,\mathrm{Re}\left(\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right). \tag{14}$$

The kernel N is continuous and the kernel M has a cotangent singularity type (see [25] for more details). Hence, the operators

$$\mathbf{N}\mu(s) := \int_J N(s,t)\mu(t)dt, \quad s \in J \tag{15}$$

is a Fredholm integral operator and the operator

$$\mathbf{M}\mu(s) := \int_J M(s,t)\mu(t)dt, \quad s \in J \tag{16}$$

is a singular integral operator.

The solvability of boundary integral equations with the generalized Neumann kernel is determined by the *index* of the function A (see [25]). The index κ_j of the function A on the curve Γ_j is defined as the change of the argument of A along the curve Γ_j divided by 2π, i.e.,

$$\kappa_j := \frac{1}{2\pi}\Delta\arg(A)|_{\Gamma_j}. \tag{17}$$

The index κ of the function A on the whole boundary curve Γ is the sum of the indexes κ_j. The index of the function A defined by (11) is given for bounded G by

$$\kappa_0 = 1, \quad \kappa_j = 0, \quad j = 1,2,\ldots,m, \quad \kappa = 1 \tag{18}$$

and for unbounded G by

$$\kappa_j = 0, \quad j = 1,2,\ldots,m, \quad \kappa = 0. \tag{19}$$

Let $\chi^{[j]}$ be the piecewise constant function defined on J by

$$\chi^{[j]}(t) := \begin{cases} 1, & \text{if } t \in J_j, \\ 0, & \text{if } t \notin J_j, \end{cases} \tag{20}$$

for $j = 1-\kappa, 2-\kappa, \ldots, m$. Then, we define the space S by

$$S = \operatorname{span}\{\chi^{[1-\kappa]}, \chi^{[2-\kappa]}, \ldots, \chi^{[m]}\}. \tag{21}$$

We define also the space $\overset{\circ}{S}$ by

$$\overset{\circ}{S} = \operatorname{span}\{\chi^{[1]}, \ldots, \chi^{[m+\kappa-1]}\}. \tag{22}$$

Then

$$\dim(S) = m + \kappa, \quad \dim(\overset{\circ}{S}) = m + \kappa - 1. \tag{23}$$

It follows from the definition of the space S that a function $h \in S$ if and only if h can be written as:

$$h(t) = \begin{cases} h_{1-\kappa}, & t \in J_{1-\kappa}, \\ h_{2-\kappa}, & t \in J_{2-\kappa}, \\ \vdots & \\ h_m, & t \in J_m, \end{cases} \tag{24}$$

with real constants $h_{1-\kappa}, h_{2-\kappa}, \ldots, h_m$. We define an operator $\mathbf{R} : S \to \overset{\circ}{S}$ for bounded G by

$$\mathbf{R}h = h - h_0$$

and for unbounded G by

$$\mathbf{R}h = h - h_m.$$

Theorem 1 ([17,19,25]). *The null-space of the operators* $\mathbf{I} \pm \mathbf{N}$ *is given by*

$$\operatorname{Null}(\mathbf{I} - \mathbf{N}) = \{0\}, \quad \operatorname{Null}(\mathbf{I} + \mathbf{N}) = S.$$

For a given function γ, it is not necessary that γ is a real part of a single-valued analytic function f on G. However, a unique piecewise constant function $\overset{\circ}{h} \in \overset{\circ}{S}$ can be obtained such that the function $\gamma + \overset{\circ}{h}$ is a real part of a single-valued analytic function (see [16, pp. 164–165]).

Theorem 2. *Let* γ *be a given function. Then, there exists a unique function* $\overset{\circ}{h} \in \overset{\circ}{S}$ *and a unique function* μ *such that*

$$f = \gamma + \overset{\circ}{h} + i\mu \tag{25}$$

are boundary values of a single-valued analytic function f in G with $\operatorname{Im} f(\alpha) = 0$ *for bounded G and* $\operatorname{Im} f(\infty) = 0$ *for unbounded G. The function* μ *is the unique solution of the integral equation*

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\gamma, \tag{26}$$

$\overset{\circ}{h} = \mathbf{R}h$, *$f(\alpha) = -h_0$ for bounded G and $f(\infty) = -h_m$ for unbounded G where h is given by*

$$h = [\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma]/2. \tag{27}$$

Proof. Let μ be the unique solution of the integral equation (26) and h is given by (27). It follows from [25, Theorem 3] that

$$Ag = \gamma + h + \mathrm{i}\mu$$

are boundary values of an analytic function g in G with $g(\infty) = 0$ for unbounded G. Let the real constant c be defined for bounded G by $c := h_0$ and for unbounded G by $c := h_m$. Then the function

$$f(z) := \Pi(z)g(z) - c$$

is analytic in G with $f(\alpha) = -c$ for bounded G, $f(\infty) = -c$ for unbounded G and has the boundary values (25) where $\overset{\circ}{h} = h - c = \mathbf{R}h \in \overset{\circ}{S}$. □

The problem of determining the single-valued analytic function f with the boundary values (25) is a special case of Riemann–Hilbert problem (see [6,10,16,23]). It is known as the *modified Dirichlet problem* [10,16] or as *Schwartz problem* [5,6].

Another possible approach for modifying the function γ so that the modified function is a real part of a single-valued analytic function in G is given in [15, p. 145]. For a given function γ and for fixed points z_j in G_j, $j = 1,2,\ldots,m$, there exists m real constants $a_1, \ldots, a_m$ such that

$$\gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j| \tag{28}$$

is a real part of a single-valued analytic function f in G with $\operatorname{Im} f(\alpha) = 0$ for bounded G and $\operatorname{Im} f(\infty) = 0$ for unbounded G. The constants $a_1, \ldots, a_m$ are uniquely determined by γ with $\sum_{j=1}^{m} a_j = 0$ for unbounded G. We shall present a method based on integral equation with the generalized Neumann kernel to calculate the real constants $a_1, \ldots, a_m$.

We define real functions $\gamma^{[j]}$ for $j = 0,1,\ldots,m$ by

$$\gamma^{[0]} := \gamma, \quad \gamma^{[j]} := \ln|\eta - z_j|, \quad j = 1,\ldots,m. \tag{29}$$

It follows from Theorem 2 that

$$f^{[j]} = \gamma^{[j]} + \overset{\circ}{h}{}^{[j]} + \mathrm{i}\mu^{[j]}, \quad j = 0,1,\ldots,m$$

are boundary values of analytic function $f^{[j]}$ in G with $\operatorname{Im} f^{[j]}(\alpha) = 0$ for bounded G and $\operatorname{Im} f^{[j]}(\infty) = 0$ for unbounded G where $\mu^{[j]}$ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu^{[j]} = -\mathbf{M}\gamma^{[j]}, \tag{30}$$

$\overset{\circ}{h}{}^{[j]} := \mathbf{R}h^{[j]}$, and $h^{[j]}$ is given by

$$h^{[j]} = [\mathbf{M}\mu^{[j]} - (\mathbf{I} - \mathbf{N})\gamma^{[j]}]/2. \tag{31}$$

Let

$$g(z) = f^{[0]}(z) + \sum_{j=1}^{m} a_j f^{[j]}(z).$$

Then g has the boundary values

$$g = \gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j| + \overset{\circ}{h}{}^{[0]} + \sum_{j=1}^{m} a_j \overset{\circ}{h}{}^{[j]} + \mathrm{i}\left(\mu^{[0]} + \sum_{j=1}^{m} a_j \mu^{[j]}\right).$$

Theorem 2 implies that $g(\alpha) = -h_0^{[0]} - \sum_{j=1}^{m} a_j h_0^{[j]}$ for bounded G and $g(\infty) = -h_m^{[0]} - \sum_{j=1}^{m} a_j h_m^{[j]}$ for unbounded G. Since the function h in Theorem 2 is unique and the function in (28) is a real part of an analytic function in G, we have

$$\overset{\circ}{h}{}^{[0]} + \sum_{j=1}^{m} a_j \overset{\circ}{h}{}^{[j]} = 0. \tag{32}$$

Hence the boundary values of the function g are given by

$$g = \gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j| + \mathrm{i}\mu,$$

where

$$\mu = \mu^{[0]} + \sum_{j=1}^{m} a_j \mu^{[j]}$$

is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\left(\gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j|\right). \tag{33}$$

It follows from (32) that the m unknowns $a_1,\ldots,a_m$ satisfy the linear equations

$$\sum_{j=1}^{m} a_j \overset{\circ}{h}{}^{[j]} = -\overset{\circ}{h}{}^{[0]}. \tag{34a}$$

Since $\overset{\circ}{h}{}^{[j]} \in \overset{\circ}{S}$, $\dim(S) = m$ for bounded G and $\dim(S) = m - 1$ for unbounded G, the system (34a) consists of m linear equations for bounded G and consists of $m - 1$ linear equations for unbounded G. However, for the unbounded case, we have the additional equation

$$\sum_{j=1}^{m} a_j = 0. \tag{34b}$$

Hence, (34) represents an $m \times m$ linear system. The existence and uniqueness of the solution of the linear system (34) follows from the existence and uniqueness of the constants $a_1, a_2, \ldots, a_m$ (see e.g. [15,16,21].) Thus we have the following theorem.

Theorem 3. *Let γ be a given function and z_j be a fixed point in G_j, $j = 1,\ldots,m$. Then, there exist m real constants $a_1,\ldots,a_m$, uniquely determined by γ, and a unique function μ such that*

$$f = \gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j| + i\mu$$

are boundary values of a single-valued analytic function f in G with $\operatorname{Im} f(\alpha) = 0$ *for bounded G and* $\operatorname{Im} f(\infty) = 0$ *for unbounded G. The constants $a_1, \ldots, a_m$ are the unique solution of the linear system (34), the function μ is the unique solution of the integral equation (33) and*

$$f(\alpha) = -h_0^{[0]} - \sum_{j=1}^{m} a_j h_0^{[j]} \tag{35}$$

for bounded G and

$$f(\infty) = -h_m^{[0]} - \sum_{j=1}^{m} a_j h_m^{[j]} \tag{36}$$

for unbounded G.

Corollary 1. *Let the boundary values of the multi-valued analytic function F in (7) be given by*

$$F = \gamma + i\mu. \tag{37}$$

Then the function μ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\gamma - \sum_{j=1}^{m} a_j \mathbf{M} \ln|\eta(t) - z_j| - \sum_{j=1}^{m} a_j (\mathbf{I} - \mathbf{N}) \arg(\eta(t) - z_j). \tag{38}$$

Proof. Since the boundary values of the multi-valued function F are given by (37), thus the boundary values of the single-valued function f in (7) are given by

$$f = \hat{\gamma} + \mathrm{i}\hat{\mu}, \tag{39}$$

where

$$\hat{\gamma} = \gamma + \sum_{j=1}^{m} a_j \ln|\eta(t) - z_j| \tag{40}$$

and

$$\hat{\mu} = \mu + \sum_{j=1}^{m} a_j \arg(\eta(t) - z_j). \tag{41}$$

Since f is single-valued and has the boundary values (39), then Theorem 2 (where $\overset{\circ}{h} = 0$) implies that

$$(\mathbf{I} - \mathbf{N})\hat{\mu} = -\mathbf{M}\hat{\gamma}$$

which in view of (40) and (41) implies that μ is the unique solution of the integral equation (38). □

4. The adjoint integral equation

For the function A defined by (11), the function $\widetilde{A}$ defined by

$$\widetilde{A}(t) = \frac{\dot{\eta}(t)}{A(t)} \tag{42}$$

is known as the "adjoint function" to the function A (see [25]). Then, the generalized Neumann kernel $\widetilde{N}$ formed with $\widetilde{A}$ is defined by

$$\widetilde{N}(s,t) := \frac{1}{\pi} \operatorname{Im}\left(\frac{\widetilde{A}(s)}{\widetilde{A}(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \tag{43}$$

We define also the real kernel $\widetilde{M}$ by

$$\widetilde{M}(s,t) := \frac{1}{\pi} \operatorname{Re}\left(\frac{\widetilde{A}(s)}{\widetilde{A}(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \tag{44}$$

Note that

$$\frac{A(t)}{A(s)} \frac{\dot{\eta}(s)}{\eta(s) - \eta(t)} = \frac{A(t)/\dot{\eta}(t)}{A(s)/\dot{\eta}(t)} \frac{\dot{\eta}(t)}{\eta(s) - \eta(t)} = -\frac{\widetilde{A}(s)}{\widetilde{A}(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)}.$$

Thus, the adjoint kernel $N^*(s,t)$ of the kernel $N(s,t)$ is related to the kernel $\widetilde{N}(s,t)$ by

$$N^*(s,t) := N(t,s) = -\widetilde{N}(s,t). \tag{45}$$

Similarly, the adjoint kernel $M^*(s,t)$ of $M(s,t)$ is related to the kernel $\widetilde{M}(s,t)$ by

$$M^*(s,t) = -\widetilde{M}(s,t). \tag{46}$$

Let the Fredholm operator $\widetilde{\mathbf{N}}$ and the singular operator $\widetilde{\mathbf{M}}$ be defined as in (15) and (16). Then (45) and (46) imply that

$$\mathbf{N}^* = -\widetilde{\mathbf{N}}, \quad \mathbf{M}^* = -\widetilde{\mathbf{M}}, \tag{47}$$

where $\mathbf{N}^*$ and $\mathbf{M}^*$ are the adjoint operators to the operators $\mathbf{N}$ and $\mathbf{M}$ respectively.

For bounded G, the index of the function A is given by (18). Thus, it follows from [25, Eq. (101)] that the space $\widetilde{R}^+ \cap \widetilde{S}^-$ in [25, Lemma 20(b)] contains only the zero function, i.e.,

$$\widetilde{R}^+ \cap \widetilde{S}^- = \{0\}. \tag{48}$$

Thus, it follows from [25, Lemma 20(b)], [25, Eq. (103)] (applied to the adjoint function $\widetilde{A}$ instead of A) and from [25, Eq. (100)] that

$$\mathrm{Null}(\mathbf{I}+\mathbf{N}) \cap \mathrm{Range}(\mathbf{I}+\mathbf{N}) = \{0\}, \tag{49}$$
$$\mathrm{Null}(\mathbf{I}+\mathbf{N}^*) \cap \mathrm{Range}(\mathbf{I}+\mathbf{N}^*) = \{0\}. \tag{50}$$

In view of (48), it follows from [25, Lemma 6], [25, Lemma 7] and [25, Lemma 19(b)] that

$$\mathrm{Null}(\mathbf{I}+\mathbf{N}^*) = \mathrm{Null}(\mathbf{M}^*). \tag{51}$$

For unbounded G, the index of the function A is given by (19). Thus, in view of the results of [19], the Eqs. (48)–(51) are also valid for unbounded G.

We define an integral operator $\mathbf{J}$ by

$$\mathbf{J}\mu(s) := \int_J \frac{1}{2\pi} \sum_{i=1-\kappa}^{m} \chi^{[i]}(s)\chi^{[i]}(t)\mu(t)dt. \tag{52}$$

Thus, we can prove that

$$\mathbf{J}^* = \mathbf{J} = \mathbf{J}^2, \quad \mathrm{Range}(\mathbf{J}) = S, \quad \mathrm{Null}(\mathbf{I}-\mathbf{J}) = S, \quad \mathrm{Null}(\mathbf{J}) = S^\perp. \tag{53}$$

Then, in view of Theorem 1 and the Fredholm alternative theorem, we have

$$\mathrm{Null}(\mathbf{J}) = \mathrm{Range}(\mathbf{I}+\mathbf{N}^*) = S^\perp. \tag{54}$$

Since $\mathrm{Range}(\mathbf{J}) = S = \mathrm{Null}(\mathbf{I}+\mathbf{N})$, thus

$$\mathbf{N}\mathbf{J} = -\mathbf{J}$$

which implies that

$$\mathbf{J}\mathbf{N}^* = \mathbf{J}^*\mathbf{N}^* = (\mathbf{N}\mathbf{J})^* = (-\mathbf{J})^* = -\mathbf{J}. \tag{55}$$

Theorem 4

$$\mathrm{Null}(\mathbf{I}+\mathbf{N}^*+\mathbf{J}) = \{0\}.$$

Proof. Let $\mu \in \mathrm{Null}(\mathbf{I}+\mathbf{N}^*+\mathbf{J})$, i.e., μ is a solution of the integral equation

$$(\mathbf{I}+\mathbf{N}^*+\mathbf{J})\mu = 0. \tag{56}$$

By multiplying (56) by $\mathbf{J}$ and using (53) and (55), we obtain $\mathbf{J}\mu = 0$ which implies that $(\mathbf{I}+\mathbf{N}^*)\mu = 0$. Thus, in view of (54), we have

$$\mu \in \mathrm{Null}(\mathbf{I}+\mathbf{N}^*) \cap \mathrm{Null}(\mathbf{J}) = \mathrm{Null}(\mathbf{I}+\mathbf{N}^*) \cap \mathrm{Range}(\mathbf{I}+\mathbf{N}^*).$$

Hence (50) implies that $\mu = 0$. □

The above theorem can also be proven by applying the approach used in proving Theorem 2 in [1].

Theorem 2 shows that the function h can be computed by means of (27) using the solution of the integral equation (26) with the generalized Neumann kernel N. We can also calculate h using an integral equation with the adjoint generalized Neumann kernel N^* as explained in the following theorem.

Theorem 5. *The function h in Theorem 2 can be written as:*

$$h = \sum_{j=1-\kappa}^{m} (\gamma, \phi^{[j]})\chi^{[j]} \tag{57}$$

where $\phi^{[j]}$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\phi^{[j]} = -\chi^{[j]}, \quad j = 1-\kappa, \ldots, m. \tag{58}$$

Proof. Since $h \in S = \text{span}\{\chi^{[1-\kappa]}, \ldots, \chi^{[m]}\}$ and $\chi^{[1-\kappa]}, \ldots, \chi^{[m]}$ are orthonormal, the function h can be written as:

$$h = \sum_{j=1-\kappa}^{m} (h, \chi^{[j]})\chi^{[j]}.$$

Let $\phi^{[j]}$ be the unique solution of (58). By multiplying (58) by $\mathbf{J}$ and using (53) and (55), we obtain

$$(\mathbf{I} + \mathbf{N}^*)\phi^{[j]} = 0, \quad \mathbf{J}\phi^{[j]} = -\chi^{[j]}.$$

Since $2h = \mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma$, we have

$$(2h, \phi^{[j]}) = (\mathbf{M}\mu, \phi^{[j]}) - ((\mathbf{I} - \mathbf{N})\gamma, \phi^{[j]}) = (\mu, \mathbf{M}^*\phi^{[j]}) - (\gamma, (\mathbf{I} - \mathbf{N}^*)\phi^{[j]}).$$

Since $(\mathbf{I} + \mathbf{N}^*)\phi^{[j]} = 0$, it follows from (51) that $\mathbf{M}^*\phi^{[j]} = 0$. Thus

$$(2h, \phi^{[j]}) = -(\gamma, 2\phi^{[j]})$$

which in view of (53) implies that

$$(h, \chi^{[j]}) = (h, -\mathbf{J}\phi^{[j]}) = -(\mathbf{J}^* h, \phi^{[j]}) = -(\mathbf{J}h, \phi^{[j]}) = -(h, \phi^{[j]}) = (\gamma, \phi^{[j]}).$$

Hence, the function h is given by (57). □

The constants $\{a_1, \ldots, a_m\}$ can be also computed using integral equation with the adjoint kernel N^* by using the same approach used in the previous theorem for computing the function h.

Theorem 6. *The derivatives γ', μ' of the function γ, μ in (37) satisfy the uniquely solvable integral equations*

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\gamma' = -\mathbf{M}^*\mu', \tag{59}$$

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\mu' = \mathbf{M}^*\gamma' + \nu, \tag{60}$$

where ν is the piecewise constant function

$$\nu(s) = a_j, \quad s \in J_j, \quad j = 1-\kappa, \ldots, m.$$

Proof. The function F' is a singled-valued analytic function in G and has the boundary values

$$\dot{\eta}(t)F'(\eta(t)) = \gamma'(t) + \mathrm{i}\mu'(t) \tag{61}$$

Let the function g be defined on G by

$$g(z) := \Pi(z)F'(z). \tag{62}$$

Then g is an analytic function in G with $g(\alpha) = 0$ for bounded G and $g(\infty) = 0$ for unbounded G and has the boundary values

$$\widetilde{A}(t)g(\eta(t)) = \gamma'(t) + \mathrm{i}\mu'(t). \tag{63}$$

Hence, it follows from [25,19] that the functions γ' and μ' satisfy the integral equations

$$(\mathbf{I} - \widetilde{\mathbf{N}})\gamma' = \widetilde{\mathbf{M}}\mu', \tag{64}$$

$$(\mathbf{I} - \widetilde{\mathbf{N}})\mu' = -\widetilde{\mathbf{M}}\gamma'. \tag{65}$$

Since

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_j} F'(\eta)d\eta = \frac{1}{2\pi\mathrm{i}} \int_{J_j} [\gamma'(t) + \mathrm{i}\mu'(t)]dt, \quad j = 1-\kappa, \ldots, m.$$

Then, by (8) and (9), and by the definitions of $\mathbf{J}$ and v, we have

$$\mathbf{J}\gamma' = 0, \tag{66}$$

$$\mathbf{J}\mu' = v. \tag{67}$$

By adding (66) to (64) and (67) to (65); and by using (47), we obtain the integral equations (59) and (60) which by Theorem 4 are uniquely solvable. □

5. The eigenvalues of the kernel *N*

In this section we shall prove that all the eigenvalues of $\mathbf{N}$ are real and belong to $[-1,1)$. The later property is very important for solving the discretizing linear system iteratively using the generalized minimum residual method GMRES [22] (see also [7]). To prove this important result, we first define the kernel $P(s,t)$ by

$$P(s,t) := \frac{1}{\pi}\mathrm{Im}\left(\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right). \tag{68}$$

The kernel P, which is special case of the generalized Neumann kernel obtained with $A = 1$, is the well-known Neumann kernel which appears frequently in the integral equations of potential theory and conformal mapping (see, e.g., [3,10,13]). The integral operator with the kernel P will be denoted by $\mathbf{P}$.

For bounded multiply connected regions G, ± 1 are eigenvalues of $\mathbf{P}$. For unbounded G, -1 is an eigenvalue and 1 is not an eigenvalue of $\mathbf{P}$ [19,25] (see also [11,13–15]). Thus, we have from [13, p. 152] (see also [14] and [11, p. 309]) the following theorem:

Theorem 7. *Let λ be an eigenvalue of $\mathbf{P}$.*

(a) If G is bounded, then $\lambda \in [-1,1]$.
(b) If G is unbounded, then $\lambda \in [-1,1)$.

In the next theorem, we shall extend the previous theorem to the case of the integral operator $\mathbf{N}$ with the generalized Neumann kernel N formed with the function A given by (11).

Theorem 8. *If λ is an eigenvalue of $\mathbf{N}$, then $\lambda \in [-1,1)$.*

Proof. For unbounded G, we have $\mathbf{N} = \mathbf{P}$. Hence Theorem 7 implies that $\lambda \in [-1,1)$. For bounded G, we have $A(t) = \eta(t) - \alpha$ and

$$\frac{A(t)}{A(s)}\frac{\dot\eta(s)}{\eta(s)-\eta(t)} = \frac{A(t)-A(s)}{A(s)}\frac{\dot\eta(s)}{\eta(s)-\eta(t)} + \frac{\dot\eta(s)}{\eta(s)-\eta(t)}.$$

Hence, we obtain

$$N^*(s,t) = P^*(s,t) - \frac{1}{\pi}\mathrm{Im}\left(\frac{\dot\eta(s)}{\eta(s)-\alpha}\right).$$

Since λ is an eigenvalue of $\mathbf{N}$, then λ is also an eigenvalue of the adjoint operator $\mathbf{N}^*$. Let ϕ be the eigenfunction to $\mathbf{N}^*$ corresponding to the eigenvalue λ, i.e.,

$$\lambda\phi(s) - \int_J N^*(s,t)\phi(t)dt = 0.$$

Thus

$$\lambda\phi(s) - \int_J P^*(s,t)\phi(t)dt + \frac{1}{\pi}Im\left(\frac{\dot\eta(s)}{\eta(s)-\alpha}\right)\int_J \phi(t)dt = 0. \tag{69}$$

It follows from Theorem 1 that the constant function $\varphi(t) := 1$ is an eigenfunction to the operator $\mathbf{N}$ corresponding to the eigenvalue -1. Thus, it follows from [13, p. 45] that ϕ is orthogonal to φ, i.e.,

$$\int_J \phi(t)dt = \int_J \phi(t)\varphi(t)dt = 0.$$

Hence (69) becomes

$$\lambda\phi(s) - \int_J P^*(s,t)\phi(t)dt = 0$$

which implies that λ is an eigenvalue of $\mathbf{P}^*$. Thus λ is an eigenvalue of $\mathbf{P}$ which, by Theorem 7, implies that $\lambda \in [-1,1]$. Since 1 is not an eigenvalue of $\mathbf{N}$ (see Theorem 1), thus $\lambda \in [-1,1)$. □

The previous theorem implies that the eigenvalue of $\mathbf{N}$ with largest absolute value is $\lambda = -1$. It follows from (50) that

$$\mathrm{Null}(\mathbf{I}-\mathbf{N})^2 = \mathrm{Null}(\mathbf{I}-\mathbf{N})$$

which implies that the geometric multiplicity of the eigenvalue $\lambda = -1$ is the same as the algebraic multiplicity. In view of Eq. (23) and Theorem 1 the multiplicity of the eigenvalue $\lambda = -1$ is $\kappa + m$.

Theorem 9 [25]. *If λ is an eigenvalue of $\mathbf{N}$ such that $\lambda \neq -1$, then $-\lambda$ is also an eigenvalue of $\mathbf{N}$.*

Theorem 10. *If λ is an eigenvalue of $\mathbf{N}^* + \mathbf{J}$, then $\lambda \in (-1, 1)$.*

Proof. Let λ be an eigenvalue of $\mathbf{N}$ and ϕ be the corresponding eigenfunction, i.e.,

$$\lambda\phi - (\mathbf{N}^* + \mathbf{J})\phi = 0. \tag{70}$$

By multiplying (70) by $\mathbf{J}$ and using (53) and (55), we obtain

$$\lambda\mathbf{J}\phi = 0.$$

Thus $\lambda = 0 \in (-1,1)$ or $\mathbf{J}\phi = 0$ which implies that

$$\lambda\phi - \mathbf{N}^*\phi = 0$$

and hence, by Theorem 8, $\lambda \in [-1,1)$. It follows from Theorem 4 that -1 is not an eigenvalue of $\mathbf{N}^* + \mathbf{J}$. Thus $\lambda \in (-1,1)$. □

It follows from the previous two theorems the following corollary.

Corollary 2

(a) If λ is an eigenvalue of $\mathbf{I}-\mathbf{N}$, then $\lambda \in (0,2]$.
(b) If λ is an eigenvalue of $\mathbf{I} + \mathbf{N}^* + \mathbf{J}$, then $\lambda \in (0,2)$.

6. The Dirichlet problem

The unique solution u of the Dirichlet problem can be calculated from the function $F(z)$ in (7) by $u(z) = \Re F(z)$. The function F will be calculated using the two methods mentioned at the end of Section 2. For both methods, we need to calculate first the values of the real constants $a_1, \ldots, a_m$. These constants can be calculated as explained in Theorem 3.

6.1. Method I

The boundary values of the function f in (7) are given by (25) where μ is the unique solution of the integral equation (26) and $\overset{\circ}{h} := \mathbf{R}h$ where h is given by (27). By the Cauchy integral formula, the function f can be calculated for $z \in G$ from

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\gamma + \overset{\circ}{h} + \mathrm{i}\mu}{\eta - z} d\eta.$$

Since $\overset{\circ}{h}$ is a piecewise constant function with $\overset{\circ}{h}(t) = 0$ for $t \in J_0$ for bounded G, it follows from the Cauchy–Goursat theorem that

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\overset{\circ}{h}}{\eta - z} d\eta = 0,$$

i.e., it is not necessary to determine the unknown function h to calculate $f(z)$ for $z \in G$ where

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\gamma + \mathrm{i}\mu}{\eta - z} d\eta.$$

By determining the values of the function $f(z)$ and the values of the real constants $a_1, \ldots, a_m$, we can calculate the values of the function $F(z)$ from (7).

6.2. Method II

The boundary values of the function F' are given by $\dot{\eta}F' = \gamma' + \mathrm{i}\mu'$ where μ' is the unique solution of the integral Eq. (60). Then, the values of function $F(z)$ can be calculated for $z \in G$ from (10). In view of (7), (35) and (36), we have

$$F(\alpha) = h_0^{[0]} - \sum_{j=1}^{m} a_j h_0^{[j]} - \sum_{j=1}^{m} a_j \log(\alpha - z_j)$$

for bounded G, and

$$F(\infty) = h_m^{[0]} - \sum_{j=1}^{m} a_j h_m^{[j]}$$

for unbounded G.

7. The Neumann problem

The unique solution u of the Neumann problem can be calculated from the function $F(z)$ in (7) by $u(z) = \Re F(z)$. The function F will be calculated using the two methods mentioned at the end of Section 2. Here, the values of the real constants $a_1, \ldots, a_m$ are known and given by [15, p. 152]

$$a_j = \int_{J_j} \gamma(t)|\dot{\eta}(t)|dt.$$

Let $\mathbf{T}(\zeta)$ be the unit tangent vector and $\mathbf{n}(\zeta)$ be the unit external normal vector to Γ at $\zeta \in \Gamma$. Let also $\theta(\zeta)$ be the angle between the normal vector $\mathbf{n}(\zeta)$ and the positive real axis, i.e., $\mathbf{n}(\zeta) = e^{\mathrm{i}\theta(\zeta)}$. Then,

$$e^{\mathrm{i}\theta(\eta(t))} = -\imath\mathbf{T}(\eta(t)) = -\mathrm{i}\frac{\dot{\eta}(t)}{|\dot{\eta}(t)|}.$$

Thus

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} = \mathrm{Re}\left[e^{\mathrm{i}\theta}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right)\right]. \tag{71}$$

Since $u(z) = \Re F(z)$, then by the Cauchy–Riemann equation, we have

$$F'(z) = \frac{\partial u(z)}{\partial x} - \mathrm{i}\frac{\partial u(z)}{\partial y}.$$

Thus

$$\mathrm{Re}\left[-\mathrm{i}\dot{\eta}F'\right] = |\dot{\eta}|\frac{\partial u}{\partial \mathbf{n}}. \tag{72}$$

Let the boundary values of the function F be given by

$$F = \psi + \mathrm{i}\phi. \tag{73}$$

Then the boundary values of the single-valued analytic function F' are given by

$$\dot{\eta}F' = \psi' + \mathrm{i}\phi'.$$

Thus the function ϕ' is known and is given by

$$\phi'(t) = \mathrm{Re}\left[-\mathrm{i}\dot{\eta}(t)F'(\eta(t))\right] = \gamma(t)|\dot{\eta}(t)|. \tag{74}$$

7.1. Method I

We can calculate the function ϕ from its derivative by

$$\phi_j(t) = b_j + \varphi_j(t)$$

where b_j is undetermined real constant and φ_j is defined by

$$\varphi_j(t) = \int_0^t \phi_j'(s)ds = \int_0^t \gamma_j(s)|\dot{\eta}_j(s)|ds.$$

Thus

$$\phi = \varphi + h,$$

where $h = (b_{1-\kappa}, \ldots, b_m) \in S$ is unknown function. Thus the boundary values of the function f in (7) are given by

$$f = \psi + \imath(\varphi + h) + \sum_{j=1}^{m} a_j \log(\eta - z_j). \tag{75}$$

We have assumed for bounded G that $\Re F(\alpha) = u(\alpha) = 0$ and $\hat{b}_0 := f(\alpha)$ is real, thus

$$\hat{b}_0 := \sum_{j=1}^{m} a_j \ln|\alpha - z_j|. \tag{76a}$$

For unbounded G, we have assumed that $\mathrm{Re}F(\infty)=0$, $\hat{b}_0 := f(\infty)$ is real and $\sum_{j=1}^{m} a_j = 0$. Thus

$$\hat{b}_0 = f(\infty) = 0. \tag{76b}$$

Let the real constant c be defined for bounded G by $c = -b_0$ and for unbounded G by $c = -b_m$ and let the function g be defined for bounded G by

$$g := -if + \mathrm{i}\hat{b}_0 + c. \tag{77}$$

Thus g is a single valued analytic function in G with $g(\alpha) = c$ for bounded G and $g(\infty) = c$ for unbounded G; and has the boundary values

$$g = \varphi + \sum_{j=1}^{m} a_j \arg(\eta - z_j) + \overset{\circ}{h} + \mathrm{i}\mu,$$

where $\overset{\circ}{h} := \mathbf{R}h$ and

$$\mu := -\psi - \sum_{j=1}^{m} a_j \ln|\eta - z_j| + \hat{b}_0$$

is unknown function. Then, Theorem 2 implies that μ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\left(\varphi + \sum_{j=1}^{m} a_j \arg(\eta - z_j)\right)$$

and $\overset{\circ}{h} = \mathbf{R}h$ where h is given by

$$h = \frac{1}{2}\left[\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\left(\varphi + \sum_{j=1}^{m} a_j \arg(\eta - z_j)\right)\right].$$

By obtaining μ, we can calculate the values of the function ψ from

$$\psi = -\mu - \sum_{j=1}^{m} a_j \ln|\eta - z_j| + \hat{b}_0.$$

Then, in view of (75) and the Cauchy integral formula, the function $f(z)$ can be calculated for $z \in G$ from

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{\psi + \imath(\varphi + h) + \sum_{j=1}^{m} a_j \log(\eta - z_j)}{\eta - z} d\eta.$$

Hence the function $F(z)$ can be calculated for $z \in G$ from (7).

7.2. Method II

In view of (73) and Eq. (59) in Theorem 6, the function ψ' is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\psi' = -\mathbf{M}^*\phi'.$$

By obtaining ψ', we obtain the boundary values of the function F', i.e., $\dot{\eta}F' = \psi' + \mathrm{i}\phi'$. Then the function F can be calculated from (10). For bounded G, we have the condition $u(\alpha) = 0$ which implies that $\mathrm{Re}F(\alpha) = 0$. For unbounded G, we have the condition $u(z) \to \infty$ for $z \to \infty$ which implies that $\mathrm{Re}F(\infty) = 0$. Since $\mathrm{Im}\, f(\alpha) = 0$ for bounded G and $\mathrm{Im}\, f(\infty) = 0$ for unbounded G, we have

$$F(\alpha) = -i\sum_{j=1}^{m} \arg(\alpha - z_j)$$

for bounded G, and

$$F(\infty) = 0$$

for unbounded G.

8. Numerical examples

Since the functions A_k and η_k are 2π-periodic, a reliable procedure for solving the integral equations (26), (59) and (60) numerically is by using the Nyström method with the trapezoidal rule [3]. Thus solving the integral equations reduces to solving a linear system

$$A\mathbf{x} = \mathbf{y}. \tag{78}$$

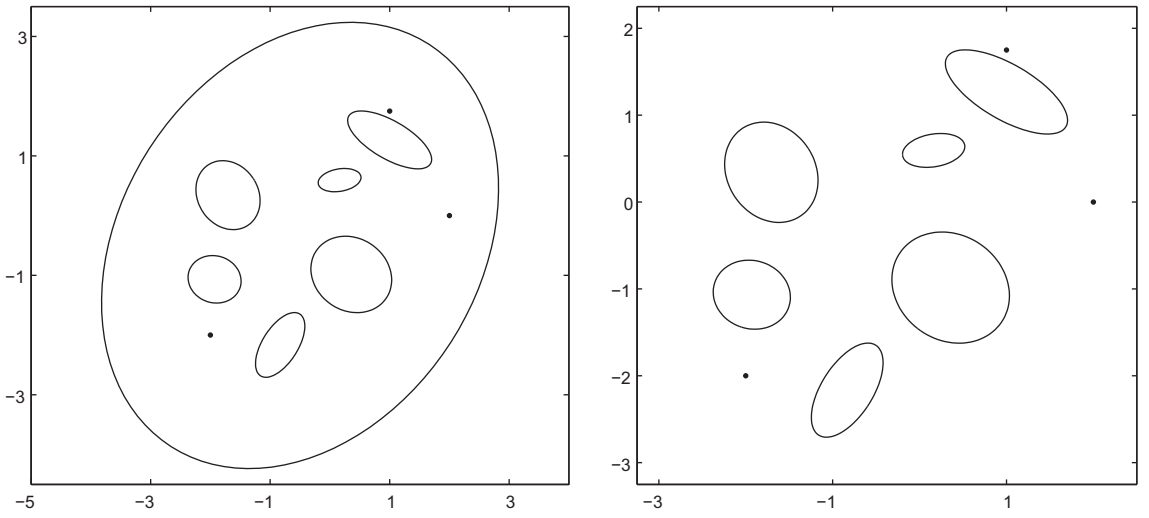

Fig. 3. The bounded region (left) and unbounded region (right).

Table 1
The values of constants a_j, b_j, z_j, θ_j and ζ_j in (79).

j	α_j	β_j	z_j	θ_j	ζ_j
0	4.0000	3.0000	−0.5000 − 0.5000i	1.0000	5.00 + 5.00i
1	0.3626	−0.1881	0.1621 + 0.5940i	3.3108	0.10 + 0.50i
2	0.5061	−0.6053	−1.7059 + 0.3423i	0.5778	−1.60 + 0.40i
3	0.6051	−0.7078	0.3577 − 0.9846i	4.1087	0.30 − 0.90i
4	0.7928	−0.3182	1.0000 + 1.2668i	2.6138	0.95 + 1.20i
5	0.3923	−0.4491	−1.9306 − 1.0663i	4.4057	−1.85 − 1.00i
6	0.2976	−0.6132	−0.8330 − 2.1650i	5.7197	−0.80 − 2.10i

Since the integral equations (26), (59) and (60) are uniquely solvable, then for sufficiently large number of collocation points on each boundary component, the linear system (78) is also uniquely solvable [3]. The linear system (78) is solved using the Gauss elimination method. The computational details are similar to previous works [17,18] in connection with numerical conformal mapping of multiply connected regions. See [20] for some ideas on how to handle regions with corners to achieve good accuracy.

In this section we consider a bounded and an unbounded multiply connected regions (see Fig. 3). These regions have been considered in [7,9]. The boundary Γ of the bounded region G is parametrized by

$$\eta_j(t) = z_j + e^{i\theta_j}(\alpha_j \cos t + i\beta_j \sin t), \quad j = 0, 1, \ldots, 6. \tag{79}$$

The values of the complex constants z_j and the real constants α_j, β_j are as in Table 1. The unbounded G is obtained by removing Γ_0.

The function γ, for the Dirichlet and Neumann problems, is obtained by choosing an exact solution of the form

$$u(z) = c + d\mathrm{Re}\left(\frac{1}{z-\zeta_0}\right) + \sum_{j=1}^{6} a_j \log(|z-\zeta_j|^2),$$

where

$$a_j = j - \frac{7}{2}, \quad j = 1, \ldots, 6,$$

$d = 2$ for bounded G and $d = 0$ for unbounded G. The constant c is given by $c = 1$ for the Dirichlet problem and $c = 0$ for the Neumann problem.

Tables 2–5 list the error $|u(z)-u_n(z)|$ at several points in G for various values of n where $u(z)$ is the exact solution of the problem, $u_n(z)$ is the approximate solution and n is the number of points used in the discretization of each boundary component Γ_j. The Dirichlet problem is solved using the first method and the Neumann problem is solved using the second method.

For numerical comparison, Tables 2–5 list also the error $|u(z) - u_n(z)|$ obtained by solving the Dirichlet problem and the Neumann problem using two classical methods. For the Dirichlet problem, we compare our method with Mikhlin's method

Table 2
The error $|u(z)-u_n(z)|$ for the Dirichlet problem for bounded G.

n	Our method			Mikhlin's method		
	$z=-2-2i$	$z=1.0+1.75i$	$z=2$	$z=-2-2i$	$z=1.0+1.75i$	$z=2$
8	5.4(−03)	2.1(−02)	5.8(−03)	5.9(−03)	4.1(−02)	8.1(−02)
16	6.8(−06)	1.0(−03)	3.3(−04)	7.5(−06)	1.1(−02)	1.7(−02)
32	1.9(−08)	1.7(−05)	3.0(−07)	7.2(−09)	4.4(−04)	3.2(−05)
64	1.4(−13)	8.6(−10)	2.1(−11)	1.9(−13)	2.6(−12)	2.1(−08)
128	2.7(−15)	5.3(−15)	1.8(−15)	3.6(−15)	4.5(−10)	8.9(−15)

Table 3
The error $|u(z)-u_n(z)|$ for the Dirichlet problem for unbounded G.

n	Our method			Mikhlin's method		
	$z=-2-2i$	$z=1.0+1.75i$	$z=2$	$z=-2-2i$	$z=1.0+1.75i$	$z=2$
8	4.4(−03)	2.4(−02)	3.7(−04)	4.0(−03)	4.4(−02)	1.4(−02)
16	9.5(−06)	1.1(−03)	1.4(−04)	1.4(−05)	1.1(−02)	1.4(−04)
32	1.5(−08)	1.7(−05)	1.0(−07)	1.5(−08)	4.4(−04)	9.6(−08)
64	4.5(−13)	8.6(−10)	1.7(−11)	4.7(−13)	2.6(−06)	1.7(−11)
128	8.9(−15)	8.0(−15)	2.7(−15)	6.2(−15)	4.5(−10)	3.6(−15)

Table 4
The error $|u(z)-u_n(z)|$ for the Neumann problem for bounded G.

n	Our method			Classical BIE		
	$z=-2-2i$	$z=1.0+1.75i$	$z=2$	$z=-2-2i$	$z=1.0+1.75i$	$z=2$
8	7.2(−02)	3.5(−01)	4.8(−01)	6.3(−01)	4.9(−01)	4.2(−01)
16	6.8(−02)	4.2(−02)	5.0(−02)	1.2(−01)	7.1(−02)	4.5(−02)
32	2.8(−05)	1.9(−04)	5.6(−05)	3.5(−07)	4.8(−04)	6.1(−06)
64	4.3(−08)	6.3(−07)	5.2(−08)	6.3(−08)	2.4(−06)	3.3(−08)
128	1.4(−14)	7.8(−12)	8.9(−15)	1.1(−14)	1.1(−10)	2.2(−16)

Table 5
The error $|u(z)-u_n(z)|$ for the Neumann problem for unbounded G.

n	Our method			Classical BIE		
	$z=-2-2i$	$z=1.0+1.75i$	$z=2$	$z=-2-2i$	$z=1.0+1.75i$	$z=2$
8	9.0(−02)	4.2(−02)	1.4(−02)	2.8(−01)	3.2(−01)	2.2(−01)
16	6.7(−03)	5.4(−03)	1.8(−02)	4.2(−02)	3.4(−02)	5.3(−03)
32	1.5(−07)	1.5(−04)	1.3(−05)	1.3(−05)	5.1(−04)	2.1(−05)
64	1.6(−09)	5.0(−07)	6.7(−09)	2.0(−08)	2.4(−06)	5.8(−09)
128	8.9(−16)	2.9(−12)	1.8(−15)	1.2(−14)	1.1(−10)	4.4(−16)

which is based on writing the solution u of the problem as a double layer potential (see [15,7,9]). Calculating the approximate solution $u_n(z)$ using our first method or Mikhlin's method requires calculating Cauchy type integral

$$\int_\Gamma \frac{\psi(\eta)}{\eta - z} d\eta, \quad z \in G. \tag{80}$$

For points z which are not close to the boundary Γ, the integrals in (80) are approximated by the trapezoidal rule. However, for points z near the boundary Γ, the integrand in (80) is nearly singular. For our method, the density function ψ is an analytic complex-valued function in G, so the integral (80) can be calculated accurately using the method suggested in [8, Eqs. (23) and (27)]. For Mikhlin's method, the density function ψ is a real-valued function and extra calculations are required to use the method described in [8, Eqs. (23) and (27)]. So, in the numerical calculations below, we calculate the integrals in (80) for Mikhlin's method using the method describe in [4, Eq. (2.17)].

For the Neumann problem, we compare our method with the classical boundary integral equation method which is based on writing the solution u of the problem as a single layer potential (see [10,7]). The integral equation is uniquely solvable for unbounded regions and non-uniquely solvable for bounded regions. However, the non-uniqueness can be removed by imposing additional conditions on the solution of the integral equation (see e.g. [1,3]).

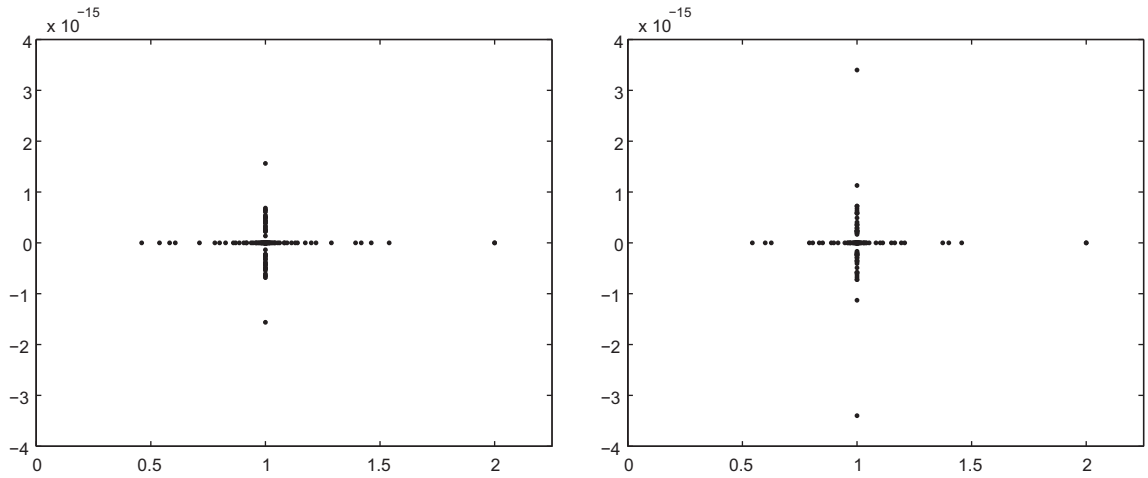

Fig. 4. The eigenvalues of the matrix A for the integral equation (26) obtained with $n = 128$ for bounded G (left) and unbounded G (right).

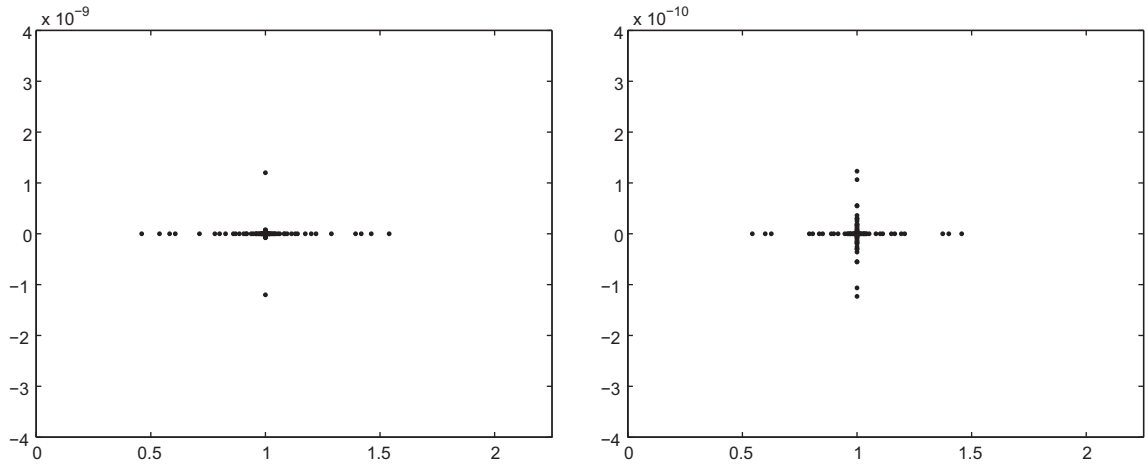

Fig. 5. The eigenvalues of the matrix A for the integral equations (59) and (60) obtained with $n = 128$ for bounded G (left) and unbounded G (right).

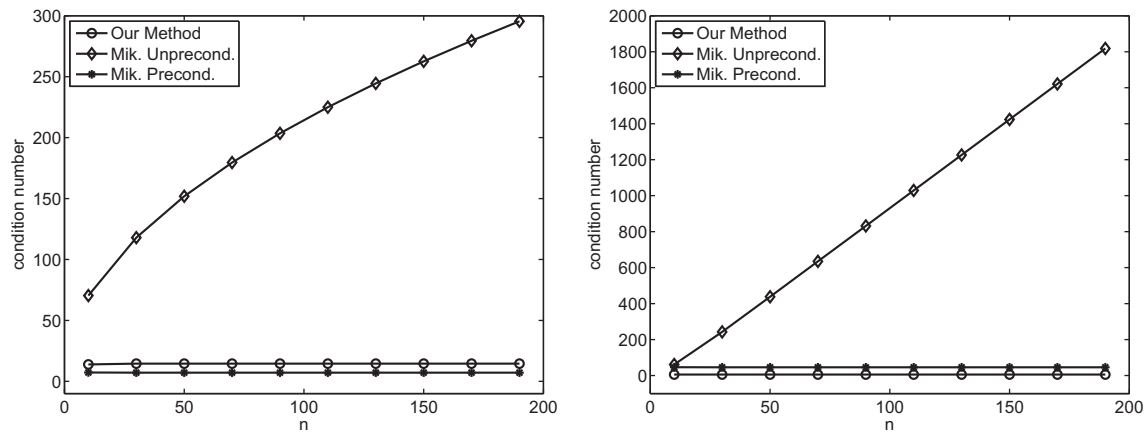

Fig. 6. The condition numbers of the matrices of the linear systems of our method (for the integral equation (26)) and Mikhlin's method (with and without preconditioning as explained in [7,9]) for bounded G (left) and unbounded G (right).

It is clear from Tables 2–4 that our method produces comparable accuracy to the classical boundary integral methods for solving the Dirichlet problem and the Neumann problem. However, our method has the following advantages:

1. The matrix of the linear system obtained by discretizing the integral equation of Mikhlin's method has in general complex eigenvalues. So, to solve the linear system iteratively, preconditioning techniques were used in [7,9]. Even the preconditioned matrices used in [7,9] have in general complex eigenvalues (see [7, Fig. 4]). However, the eigenvalues of the matrices of the linear system obtained by discretizing our integral equations are real. For sufficiently large number of collocation points on each boundary component, in view of Corollary 2, the eigenvalues of the matrix A are positive real numbers in the interval $(0,2]$ for the integral equation (26) and in the interval $(0,2)$ for the integral equations (59) and (60) (see e.g. [2,3]). For both cases, the eigenvalues are real and clustered around 1 (see Figs. 4 and 5). The latter property means that iterative methods will converge for our linear system faster than the preconditioned and unpreconditioned linear systems in [7,9].
2. A comparison between the condition numbers of the matrices of the linear systems of our method (for the integral equation (26)) and Mikhlin's method (with and without preconditioning as explained in [7,9]) for various values of n are given in Fig. 6. As can be seen from the figure, the condition numbers of the matrix of our method and the preconditioned matrix of Mikhlin's method is independent of n. However, the condition number of the unpreconditioned matrix of Mikhlin's method depends on n.

9. Conclusions

In this paper we have presented two uniquely solvable boundary integral equations for solving Laplace's equation with the Dirichlet boundary condition or the Neumann condition on both bounded and unbounded regions. The integral equations are second kind Fredholm integral equations with the generalized Neumann kernel which has been derived and studied in [17–20,24,25].

To illustrate the accuracy of the presented methods, we solve the Dirichlet problem and the Neumann problem on a bounded and an unbounded multiply connected regions. The integral equations are solved numerically by the Nyström method with the trapezoidal rule. The presented numerical results illustrate that the proposed method can be used to produce approximations of high accuracy. We presented also numerical comparison between our method and Mikhlin's method which is a classical boundary integral method for solving the Dirichlet problem and the Neumann problem [15,7,9].

An efficient method for solving the linear system obtained by discretizing the integral equation of Mikhlin's method has been presented in [7,9] where the linear system is solved by GMRES iterative method powered by the Fast Multiple Method (FMM). Solving the linear system obtained by discretizing our integral equations by such an efficient iterative method is certainly recommended when the connectivity of the region assumes much larger value or when the boundary components Γ_j lie closed to each other where more discretization points are needed. It is worth mentioning that due to the properties of the generalized Neumann kernels, our integral equations has some advantages over the integral equation of Mikhlin's method. The matrix of the linear system obtained by discretizing the integral equation of Mikhlin's method has in general complex eigenvalues (see [7, Fig. 4]). However, the eigenvalues of the matrices of the linear systems obtained by discretizing our integral equations are positive real numbers clustered around 1, belong to the interval $(0,2]$ for the integral equation (26) and belong to the interval $(0,2)$ for the integral equations (59), (60) (see Figs. 4 and 5.) In view of [22, p. 866], the latter property means that the GMRES method will converge for our linear system faster than for the linear systems in [7,9].

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