A Novel Analytical Method Predicts Plug Boundaries of Bingham Plastic Fluids for Laminar Flow Through Annulus

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In Bingham plastic fluids, a central pre-yield or plug region exists in the middle of the concentric annular flow. In this region, the local shear stress is less than the dynamic yield stress, so the plug behaves like a rigid solid. This is the main feature that distinguishes the flow of a Bingham plastic fluid from that of a power law fluid. In this work, a simple-to-use correlation is developed to predict the boundaries of the plug of Bingham plastic fluids for laminar flow through annulus as a function of dimensionless yield stress and aspect ratio parameters for given values of the rheological constants, pressure gradient and the dimensions of the annulus. The results are found to be in excellent agreement with reported data in the literature with an average absolute deviation of less than 1.7%.

The predictive tool is simple and straightforward and can be readily implemented in a standard spreadsheet program. The prime application of the method is as a quick-and-easy evaluation tool in engineering studies where plug boundaries of Bingham plastic fluids for laminar flow through annulus are being considered. The method may also serve as a benchmark in numerical and rigorous simulation studies.

Keywords: non-Newtonian fluid, Bingham plastic fluid, concentric annulus, Vandermonde matrix, predictive tool, yields stress, aspect ratio parameter

INTRODUCTION

A Bingham plastic fluid has a yield point, which is the shear stress that has to be overcome so that the fluid can start to flow.[1−4] The main feature that distinguishes the flow of a Bingham plastic fluid from that of a power law fluid is the existence of a plug region in which the shear stress is less than the yield stress,[5−8] which means a fluid with a yield stress will flow only if the applied stress (proportional to pressure gradient) exceeds the yield stress.[9−12] Therefore, there will be a solid plug-like core flowing in the middle of the pipe where shear stress is less than the yield stress.

Extensive works on non-Newtonian fluid mechanics and annular flows have been presented in the literature.[13−17] The flow of viscoplastic materials is present in a large number of industrial processes. The large variety of fluids and industrial applications of annular flows of non-Newtonian fluids has been a major motivation for research in annular flow with varying degrees of complexity.[18,19] One important example is sterilisation or packaging processes of foods, pharmaceutics products, cosmetics and lubricants, the drilling process of oil wells and the extrusion of ceramic catalyst supports.[20] In the petroleum industry, the drilling muds are typically either Bingham plastic or power-law type fluids. Other examples include the extrusion of plastic tubes and pipes in which the molten polymer is forced through an annulus and the flow in double-pipe heat exchangers.[21−23]

The laminar axial flow of Bingham plastic fluids through a concentric annulus has generated great interest, even more than power-law fluids in non-Newtonian fluid mechanics. The calculation of the velocity distribution and the mean velocity of a fluid flowing through an annulus of outer radius 'R' and inner radius 'σR' (Figure 1) is more complex than that for flow in a pipe or between two parallel planes.[24,25] In this work, the proposed method is suitable in conceptual development and scoping studies where the estimation of the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) are being considered. This gives us the confidence to offer our findings for engineering applications where a rough and ready programmable estimate is sought.

Mathematical Modelling

In Figure 1, consider the flow of the fluid situated at a distance not greater than r from the centre-line of the pipe. The shear force acting on this fluid comprises two parts: one is the drag on its outer surface ($r = R$), which can be expressed in terms of the shear stress in the fluid at that location; the other contribution is the drag occurring at the inner (solid) boundary of the annulus, that is at $r = \sigma R$. This component cannot be estimated at present; however, alternatively, this difficulty can be obviated by considering the equilibrium of a thin ring of fluid of radius r and thickness dr (Figure 1). The pressure force acting on this fluid element is[24,25]:

$$2\pi r\, \mathrm{d}r(p - (p + \Delta p)) \tag{1}$$

The only other force acting on the fluid element in the z-direction is that arising from the shearing on both surfaces of the element. Note that, not only will the shear stress change from r to $r + \mathrm{d}r$, but the surface area over which shearing occurs will also depend upon the value of r. The net force can be written as:

$$2\pi r L \tau_{rz}|_{r+\mathrm{d}r} - 2\pi r L \tau_{rz}|_{r} \tag{2}$$

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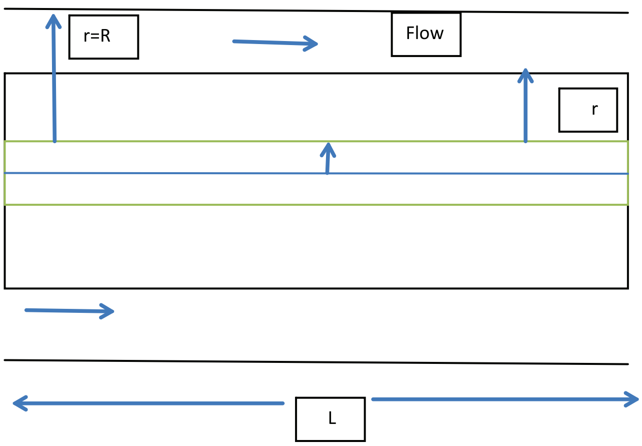

Figure 1. Flow in a concentric annulus.

At equilibrium, therefore:

$$2\pi r\,\mathrm{d}r\left(\frac{-\Delta p}{L}\right)L = 2\pi L\left(r\tau_{rz}|_{r+\mathrm{d}r} - r\tau_{rz}|_{r}\right) \tag{3}$$

or:

$$\frac{r\tau_{rz}|_{r+\mathrm{d}r} - r\tau_{rz}|_{r}}{\mathrm{d}r} = r\left(\frac{-\Delta p}{L}\right) \tag{4}$$

Now taking limits as $\mathrm{d}r \to 0$, it becomes:

$$\frac{d}{\mathrm{d}r}(r\tau_{rz}) = r\left(\frac{-\Delta p}{L}\right) \tag{5}$$

The shear stress distribution across the gap is obtained by integration:

$$\tau_{rz} = \frac{r}{2}\left(\frac{-\Delta p}{L}\right) + \frac{C_1}{r} \tag{6}$$

where C_1 is a constant of integration. Because of the no-slip boundary condition at both solid walls, that is at $r=\sigma R$ and $r=R$, the velocity must be maximum at some intermediate point, say at $r=\lambda R$. Then, for a fluid without a yield stress, the shear stress must be zero at this position. Thus, the constant of integration, C_1, can be evaluated using the condition of $\tau_{rz}=0$ at $r=\lambda R$, and hence:

$$C_1 = -\frac{\lambda^2 R^2}{2}\left(\frac{-\Delta p}{L}\right) \tag{7}$$

and this allows Equation (6) to be rewritten as:

$$\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}\left(\xi - \frac{\lambda^2}{\xi}\right) \tag{8}$$

where $\xi = r/R$, is the dimensionless radial coordinate.

In principle, the velocity distribution and the mean velocity of a Bingham plastic fluid flowing through an annulus can be deduced by substituting for the shear stress in Equation (8) in terms of the Bingham plastic model. Figure 2 shows qualitatively the salient features of the velocity distribution in an annulus for Bingham plastic fluid. However, the signs of the shear stress (considered positive in the same sense as the flow) and the velocity gradients in the two flow regions need to be treated with special care. The

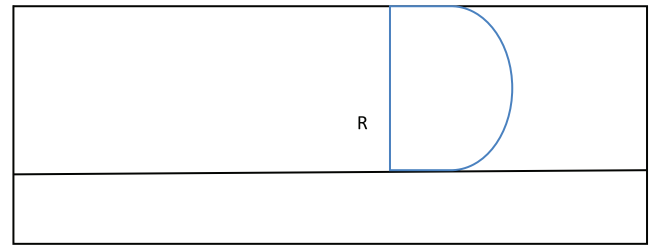

Figure 2. Schematics of velocity profile.

shearing force on the fluid is positive $(\sigma R \le r \le \lambda - R)$ where the velocity gradient is also positive. Thus, in this region[24,25]:

$$\tau_{rz} = \tau_0^{\mathrm{B}} + \mu_{\mathrm{B}}\left(\frac{\mathrm{d}V_Z}{\mathrm{d}r}\right) \tag{9}$$

On the other hand, in the region $\lambda + R \le r \le R$, the velocity gradient is negative and the shearing force is also in the negative r-direction, and hence:

$$-\tau_{rz} = \tau_0^{\mathrm{B}} + \mu_{\mathrm{B}}\left(-\frac{\mathrm{d}V_Z}{\mathrm{d}r}\right) \tag{10}$$

Equations (9) and (10) can now be substituted in Equation (8) and integrated to deduce the velocity distributions. The constants of integration can be evaluated by using the no-slip boundary condition at both $r=\sigma R$ and $r=R$. However, the boundaries of the plug existing in the middle of the annulus are not yet known; nor is the plug velocity known. The unknown boundaries are evaluated by applying the following conditions: the continuity of velocity at $r=\eta - R$ and $r=\lambda + R$ (the velocity gradient is also zero at these boundaries), and the force balance on the plug of fluid is[24,25]:

$$2\pi R(\lambda+\eta)\tau_0^B = \left(-\frac{\Delta P}{L}\right)\pi\left[(\lambda+R)^2 - (\lambda-\eta)^2\right] \tag{11}$$

The algebraic steps required to carry out the necessary integrations and the evaluation of the constants are quite involved and tedious. Thus, these are not presented here, and readers are referred to the original papers (Laird, 1957) or to the book by Skelland[25] for detailed derivations. Instead, consideration is given here to the practical problem of estimating the necessary pressure gradient to maintain a fixed flow rate of a Bingham plastic fluid or vice versa. Fredrickson and Bird[20] organised their numerical solutions of the equations presented above in terms of the following dimensionless parameters:

$$\text{Dimensionless velocity}: V_Z^* = \frac{2\,\mu_{\mathrm{B}}\,V_Z}{R^2((-\Delta P)/L)} \tag{12}$$

$$\text{Dimensionless yield stress}: \phi_0 = \frac{2\tau_0^{\mathrm{B}}}{R((-\Delta P)/L)} \tag{13}$$

$$\text{Dimensionless flow rate}: \Omega = \frac{Q}{Q_{\mathrm{N}}} \tag{14}$$

where Q_N is the flow rate of a Newtonian liquid of viscosity, μ_B, and thus:

$$Q_N = \frac{\pi R^4}{8\ \mu_B}\left(\frac{-\Delta p}{L}\right) \tag{15}$$

In view of the above, in this work, the boundaries (λ_+) of plug for Bingham plastic fluids laminar flow through an annulus is correlated as a function of dimensionless yield stress and aspect ratio parameter for given values of the rheological constants, pressure gradient and the dimensions of the annulus. This paper discusses the formulation of such a predictive tool in a systematic manner along with an example to show the simplicity of the method and usefulness of such a tool. The proposed method is an exponential function, which leads to well-behaved (i.e. smooth and non-oscillatory) equations enabling more accurate and non-oscillatory predictions. This is the distinct advantage of the proposed method.

Fredrickson and Bird[20] presented data showing relationships between ϕ_0, Ω and σ. In this work, for given values of the rheological constants (μ_B, τ_0^B) and the dimensions of the annulus (σ, R), the boundaries of plug (λ) for Bingham plastic fluids laminar flow through an annulus is calculated as a function of dimensionless yield stress and aspect ratio parameter.

METHODOLOGY FOR THE DEVELOPMENT OF NOVEL PREDICTIVE TOOL

The primary purpose of the present study is to accurately correlate the boundaries of plug (λ) for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ).

In this work, the Vandermonde matrix is used to adjust the parameters. The denials of the Vandermonde matrix is reported in the appendix of this paper.

The required data[20,24] to develop this correlation include the values of the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ).

The following methodology[26–29] using Matlab[30] has been applied to develop this correlation. Firstly, data values of the boundaries of plug for Bingham plastic fluids through an annulus[20,24] are correlated as a function of aspect ratio parameter (σ) for different dimensionless yield stress (ϕ_0) values, then the calculated coefficients for these equations are correlated as a function of dimensionless yield stress (ϕ_0). The derived equations are applied to calculate new coefficients for Equation (16) to predict the boundaries of plug for Bingham plastic fluids through an annulus.

Table 1 shows the tuned coefficients for Equations (17)–(20) to predict the boundaries of plug (λ) for Bingham plastic fluids through an annulus.[20,24]

In brief, the following steps are repeated to tune the correlation's coefficients using Matlab[30]:

(1) Correlate the boundaries of plug (λ) for Bingham plastic fluids through an annulus as a function aspect ratio parameter (σ) for a given dimensionless yield stress (ϕ_0) value.
(2) Repeat step 1 for other dimensionless yield stress (ϕ_0) data.
(3) Correlate corresponding polynomial coefficients, which were obtained for dimensionless yield stress (ϕ_0) data versus 'ϕ_0' parameter, $a=f(\phi_0)$, $b=f(\phi_0)$, $c=f(\sigma\phi_0)$, $d=f(\phi_0)$ (Equations 17–20).

Table 1. Tuned coefficients used in Equations (17)–(20)

Coefficient	Value of dimensionless yield stress (ϕ_0) less than 0.5	Value of dimensionless yield stress (ϕ_0) greater than 0.5
A_1	−1.150276192	−2.175571164
B_1	$1.475517695 \times 10^{-1}$	5.953316650
C_1	4.051711092	−7.242934287
D_1	−4.008893104	3.540015028
A_2	3.283658498	2.256507132
B_2	7.724316435	−2.516165737
C_2	$-3.077839119 \times 10^{1}$	$1.444928294 \times 10^{1}$
D_2	$2.661170364 \times 10^{1}$	$-1.463844785 \times 10^{1}$
A_3	−4.098751522	$7.299848912 \times 10^{1}$
B_3	$-2.038451497 \times 10^{1}$	$-3.007070405 \times 10^{2}$
C_3	$5.533689317 \times 10^{1}$	$3.365693174 \times 10^{2}$
D_3	$-4.584582868 \times 10^{1}$	$-1.024886714 \times 10^{2}$
A_4	1.978128142	$-3.663618770 \times 10^{2}$
B_4	1.43715826×10^{1}	$1.595992468 \times 10^{3}$
C_4	$-2.925170868 \times 10^{1}$	$-2.185947051 \times 10^{3}$
D_4	$3.164537814 \times 10^{1}$	$9.600108170 \times 10^{2}$

Equation (16) represents the proposed governing equation in which four coefficients are used to correlate the boundaries of plug (λ) for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ). The relevant coefficients have been reported in Table 1

$$\ln(\lambda) = a + b\sigma + c\sigma^2 + d\sigma^3 \tag{16}$$

where:

$$a = A_1 + B_1\phi_0 + C_1\phi_0^2 + D_1\phi_0^3 \tag{17}$$

$$b = A_2 + B_2\phi_0 + C_2\phi_0^2 + D_2\phi_0^3 \tag{18}$$

$$c = A_3 + B_3\phi_0 + C_3\phi_0^2 + D_3\phi_0^3 \tag{19}$$

$$d = A_4 + B_4\phi_0 + C_4\phi_0^2 + D_4\phi_0^3 \tag{20}$$

The optimum tuned coefficients given in Table 1 can be further retuned quickly according to the proposed approach if more data become available in the future. Figure 3 shows the flow chart to adjust the tuned coefficients for Equations 17–20.

In this work, our efforts were directed at formulating a correlation which can be expected to assist engineers for rapid calculation of the boundaries of plug (λ) for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ). The proposed novel tool developed in the present work is a simple and unique expression, which is non-existent in the literature. Furthermore, the selected exponential function to develop the tool leads to well-behaved (i.e. smooth and non-oscillatory) equations enabling reliable and more accurate predictions.

RESULTS

Figures 4 and 5 show the proposed method's results to estimate the boundaries of plug (λ) for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ). Figure 6 shows the smoothness of predictive tool's results in the prediction of the boundaries of

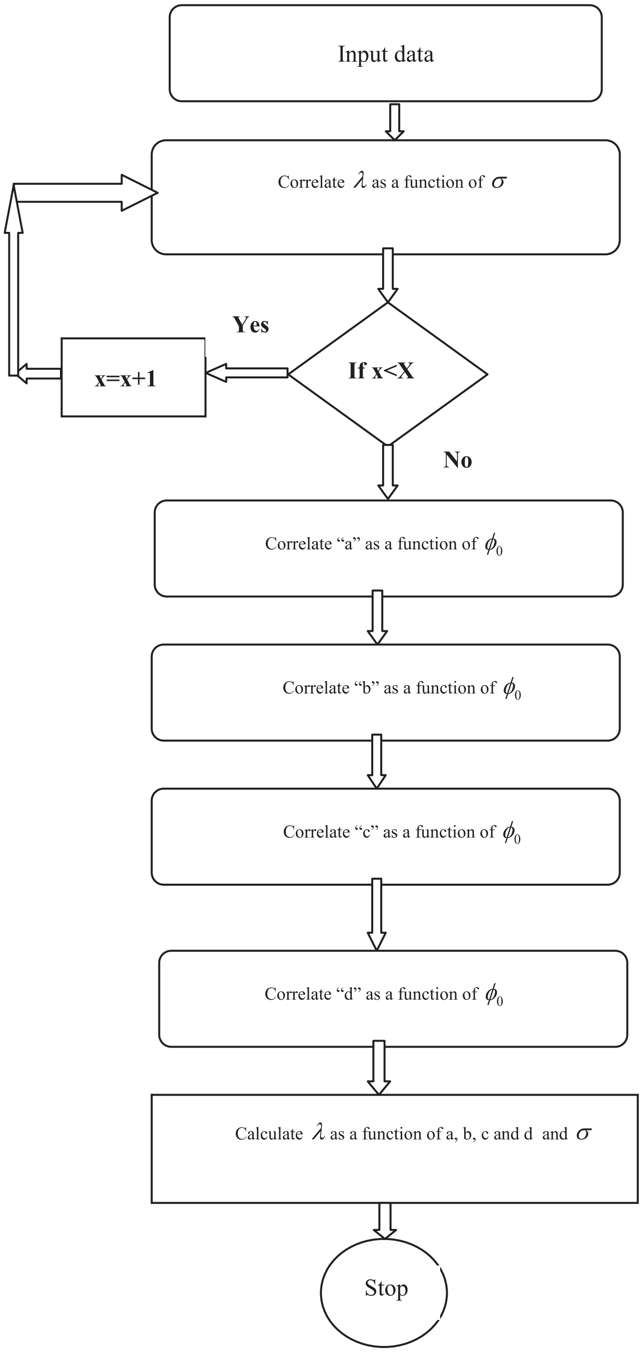

Figure 3. General workflow of algorithm used for tuning the coefficients in Equations (17)–(20).

plug for Bingham plastic fluids laminar flow through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ).

Figure 7 illustrates a parity chart to illustrate the excellent performance of the method. Table 2 illustrates the accuracy of proposed correlation for predicting the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) in comparison with some reported data.[20,24]

The deviation of correlation in terms of average absolute deviation is around 1.7%. It is expected that our efforts in formulating a simple tool will pave the way for arriving at an accurate

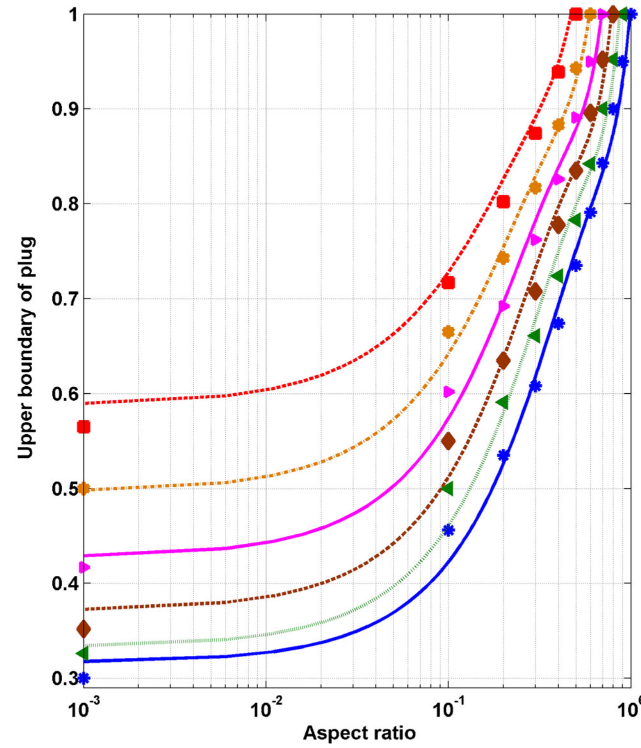

Figure 4. The performance of predictive tool in comparison with data[20,24] to calculate the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) dimensionless yield stress (ϕ_0) less than 0.5.

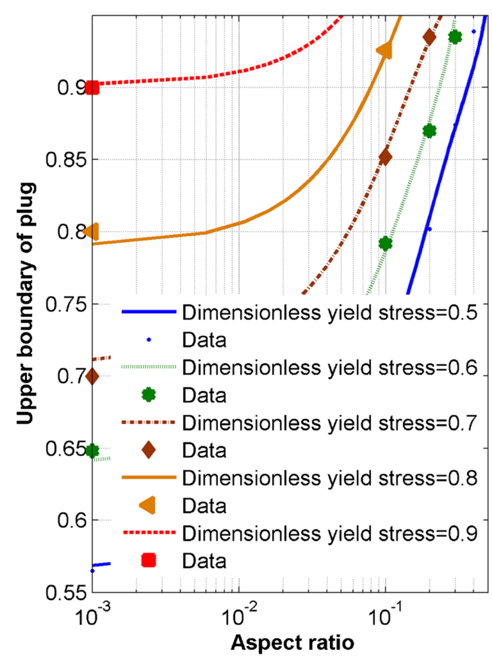

Figure 5. The performance of the predictive tool in comparison with data[20,24] to calculate the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) dimensionless yield stress (ϕ_0) greater than 0.4.

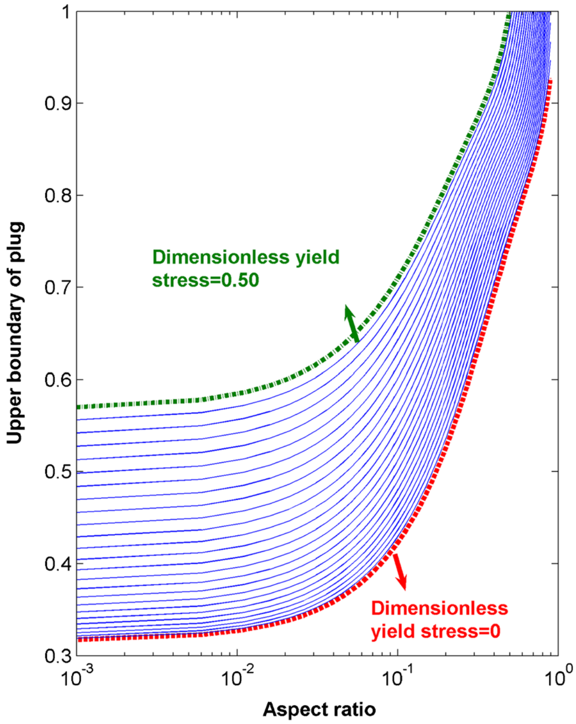

Figure 6. The smooth performance of the predictive tool to the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) dimensionless yield stress (ϕ_0) less than 0.5.

prediction of the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ), which can be used by engineers for monitoring the key parameters periodically.

The tool developed in this study can be of immense practical value for experts and engineers to have a quick check on prediction of the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ) without opting for any experimental trials. In particular, engineers would find the approach to be user-friendly with transparent calculations involving no complex expressions. Sample calculations[20,24] shown below clearly demonstrate the simplicity of the proposed tool and the benefits associated with such estimations.

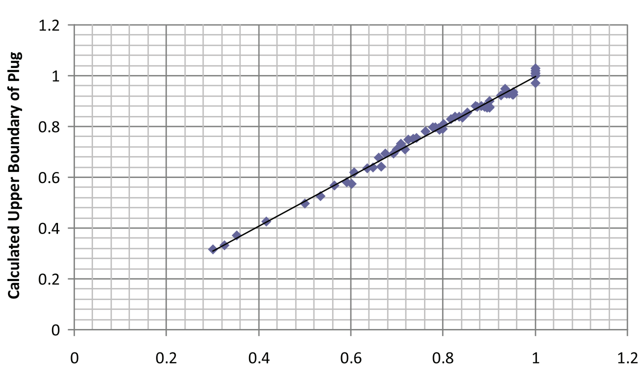

Figure 7. Parity chart to illustrate the accuracy of the predictive tool in predicting the boundaries of plug for Bingham plastic fluids flow through an annulus.

Table 2. Comparison of calculated values with typical data[20,24]

Aspect ratio	Value of dimensionless yield stress	Reported the boundaries of plug for Bingham fluid data	Calculated the boundaries of plug for Bingham fluid	Absolute deviation percent
0	0	0.3	0.316	5.3
0.2	0	0.535	0.526	1.7
0.3	0	0.608	0.618	1.6
0.4	0	0.674	0.693	2.8
0.5	0	0.735	0.751	2.2
0.6	0	0.791	0.795	0.5
0.7	0	0.843	0.834	1.1
0.8	0	0.9	0.875	2.8
0.9	0	0.95	0.93	2.1
1	0	1	1.01	1
0	0.1	0.326	0.333	2.1
0.2	0.1	0.591	0.580	1.9
0.3	0.1	0.661	0.678	2.6
0.4	0.1	0.724	0.749	3.5
0.5	0.1	0.7826	0.798	2
0.6	0.1	0.842	0.835	0.8
0.7	0.1	0.9	0.876	2.7
0.8	0.1	0.952	0.935	1.8
0.9	0.1	1	1.03	3
0	0.2	0.352	0.371	5.4
0.2	0.2	0.635	0.637	0.3
0.3	0.2	0.708	0.732	3.4
0.4	0.2	0.778	0.796	2.3
0.5	0.2	0.835	0.838	0.4
0.6	0.2	0.896	0.875	2.3
0.7	0.2	0.952	0.927	2.6
0.8	0.2	1	1.02	2
0	0.3	0.417	0.427	2.4
0.1	0.3	0.602	0.574	4.7
0.2	0.3	0.692	0.695	0.4
0.3	0.3	0.762	0.782	2.6
0.4	0.3	0.826	0.838	1.5
0.5	0.3	0.891	0.879	1.3
0.6	0.3	0.95	0.927	2.4
0.7	0.3	1	1.01	1
0	0.4	0.5	0.496	0.8
0.1	0.4	0.665	0.642	3.5
0.2	0.4	0.743	0.754	1.5
0.3	0.4	0.817	0.829	1.5
0.4	0.4	0.883	0.880	0.3
0.5	0.4	0.943	0.929	1.5
0.6	0.4	1	1	0
0	0.5	0.565	0.568	0.5
0.1	0.5	0.717	0.709	1.1
0.2	0.5	0.802	0.81	1
0.3	0.5	0.874	0.876	0.2
0.4	0.5	0.939	0.930	1
0.5	0.5	1	1	0
0	0.6	0.648	0.640	1.2
0.1	0.6	0.792	0.786	0.8
0.2	0.6	0.87	0.88	1.1
0.3	0.6	0.935	0.949	1.5
0	0.7	0.7	0.709	1.3
0.1	0.7	0.852	0.855	0.4
0.2	0.7	0.935	0.933	0.2
0.3	0.7	1	0.97	3
0	0.8	0.8	0.789	1.4
0.1	0.8	0.926	0.922	0.4
0.2	0.8	1	1.00	0
0	0.9	0.9	0.90	0
0.1	0.9	1	1	0
Average absolute deviation percent 1.65				

Example

A molten chocolate (density = 1500 kg/m^3) flows through a concentric annulus of inner and outer radii 10 and 20 mm, respectively, at 30°C at the constant flow rate of 0.03 m^3/min. The steady-shear behaviour of the chocolate can be approximated by a Bingham plastic model with $\tau_0^B = 35$ Pa and $\mu_B = 1$ Pa s, $\phi_0 = 0.048$. Determine the size of the plug.

Solution

$\tau_0^B = 35$ Pa and $\mu_B = 1$ Pa s, $\sigma = (10/20) = 0.5$, $R = 20 \times 10^{-3}$ m and $Q = 0.03$ m^3/min = (0.03/60) m^3/s.

We can now calculate pressure gradient for $\phi_0 = 0.048$

$$\left(\frac{-\Delta P}{L}\right) = \frac{2\tau_0^B}{R\phi_0} = \frac{2 \times 35}{20 \times 10^{-3} \times 0.048} = 73\,000\ \text{Pa/m} = 73\ \text{kPa/m}$$

From new proposed correlation. For $\phi_0 = 0.048$ and $\sigma = 0.50$:

1. $a = -1.134301$ (from Equation 17).
2. $b = 3.586455$ (from Equation 18).
3. $c = -4.95478$ (from Equation 19).
4. $d = 2.604067$ (from Equation 20).
5. $\lambda = 0.76$ (from Equation 16).

From the definition of λ from Equation (11), we have:

$$(\lambda - \eta)\frac{R}{2}\left(\frac{-\Delta P}{L}\right) = \tau_0^B$$

Substitution of values of λ, R, τ_0^B and $(-(\Delta P)/L)$ gives $\eta = 0.71$. Thus, the plug region extends from $R - \eta$ to $R + \lambda$, that is from 14.2 to 15.2 mm. These calculations assume the flow to be laminar. As a first approximation, one can define the corresponding Reynolds number based on the hydraulic diameter, D_h

$$D_h = \frac{4 \times \text{flow rate}}{\text{wetted perimeter}} = \frac{4\pi R^2(1-\sigma^2)}{2\pi R(1+\sigma)} = 2R(1-\sigma)$$

$$= 2 \times 20 \times 10^{-3}(1 - 0.5) = 0.02\ \text{m},$$

$$\text{Reynolds number } (Re) = \frac{\rho V D_h}{\mu_B} = \frac{1500 \times 0.53 \times 0.02}{1} = 16$$

Thus, the flow is laminar and streamlined.

CONCLUSIONS

In this work, simple-to-use equations are presented here for the estimation of the boundaries of plug for Bingham plastic fluids through an annulus as a function of dimensionless yield stress (ϕ_0) and aspect ratio parameter (σ). Unlike complex mathematical approaches, the proposed correlation is simple-to-use and would be of immense help for engineers, especially those dealing with fluids flow in concentric annulus.

Additionally, the level of mathematical formulations associated with the estimation of the boundaries of plug for Bingham plastic fluids through an annulus can be easily handled by an engineer or practitioner without any in-depth mathematical abilities. Furthermore, estimations are quite accurate as evidenced from the comparisons with literature data[20,24] (with average absolute deviation being around 1.7%). The proposed method has clear numerical background, wherein the relevant coefficients can be retuned quickly if more data become available in the future.

NOMENCLATURE

A	tuned parameter
B	tuned parameter
C	tuned parameter
D	tuned parameter
i	index
j	index
m	power law consistency coefficient
n	power-law flow behaviour index
p	pressure (N/m^2)
Q	volumetric flow rate (m^3/s)
R	pipe radius/annulus outer radius (m)
r	radial coordinate (m)
u	coefficient of polynomial
V	Vandermonde matrix
V_z	point velocity of flow in z-direction (m/s)
V_Z^*	non-dimensional point velocity
x	data point
X	maximum data point
y	distance from wall (m)
z	axial coordination in flow direction (m)

Greek Symbols

η	lower boundary of plug of Bingham plastic in annular flow
λ	upper boundary of plug of Bingham plastic in annular flow
μ_B	Bingham plastic viscosity (Pa s)
ξ	non-dimensional radial coordinate
σ	non-dimensional inner radius of annulus
τ	shear stress in fluid (N/m^2)
τ_0^B	yield stress in Bingham plastic model (Pa)
τ_{rz}	shear stress in fluid (Pa)
ϕ_0	non-dimensional stress ratio
Ω	non-dimensional flow rate

APPENDIX

The Vandemonde matrix is a matrix with the terms of a geometric progression in each row, that is an $m \times n$ matrix[31–33]

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{n-1} \end{bmatrix} \quad (A.1)$$

or

$$V_{i,j} = \alpha_i^{j-1} \quad (A.2)$$

For all indices i and j, the determinant of a square Vandermonde matrix (where $m = n$) can be expressed as[31–33]:

$$\det(V) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i) \quad (A.3)$$

The Vandermonde matrix evaluates a polynomial at a set of points; formally, it transforms coefficients of a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ to the values the polynomial takes at the point's α_i. The non-vanishing of the Vandermonde determinant for distinct points α_i shows that, for distinct points, the map from coefficients to values at those points is a one-to-one correspondence, and thus that the polynomial interpolation problem is solvable with unique solution: this result is called the unisolvence theorem[31–33]

They are thus useful in polynomial interpolation, since solving the system of linear equations $Vu = y$ for u with V and $m \times n$ Vandermonde matrix is equivalent to finding the coefficients u_j of the polynomial(s)[31–33]:

$$P(x) = \sum_{j=0}^{n-1} u_j x^j \tag{A.4}$$

For degree $\leq n-1$ which has (have) the property:

$$P(\alpha_i) = y_i \text{ for } i = 1, \ldots, m \tag{A.5}$$

The Vandermonde matrix can easily be inverted in terms of Lagrange basis polynomials: each column is the coefficients of the Lagrange basis polynomial, with terms in increasing order going down. The resulting solution to the interpolation problem is called the Lagrange polynomial[31–33]:

$$P'(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \tag{A.6}$$

The statement that P' interpolates the data points means that:

$$P'(x_i) = y_i \quad \text{forall} \quad i \in \{0, 1, \ldots, n\} \tag{A.7}$$

If we substitute Equation (1) in here, we get a system of linear equations in the coefficients a_k. The system in matrix–vector form reads:

$$V = \begin{bmatrix} x_0^n & x_0^{n-1} & x_0^{n-2} & \cdots & x_0 & 1 \\ x_1^n & x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_n^n & x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{A.8}$$

This system is solved for a_k to construct the interpolant $p(x)$. The matrix on the left is commonly referred to as a Vandermonde matrix.[31–33]

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