CONJUGACY CLASSES AND GRAPH OF 2-GROUPS OF NILPOTENCY CLASS TWO

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Dedicated to my

beloved husband, Jagan

two lovely kids, Shamita Aradhana and Sanjaay Dev,

and

parents

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ABSTRACT

Two elements a and b of a group are called conjugate if there exists an element g in the group such that $gag^{-1} = b$. The set of all conjugates in a group forms the conjugacy classes of the group. The main objective of this research is to determine the number and size of conjugacy classes for 2-generator 2-groups of nilpotency class two. Suppose G is a 2-generator 2-group of class two which comprises of three types, namely Type 1, Type 2 and Type 3. The general formulas for the number of conjugacy classes of G are determined by using the base group and central extension method, respectively. It is found that for each type of the group G, the number of conjugacy classes consists of two general formulas. Moreover, the conjugacy class sizes are computed based on the order of the derived subgroup. The results are then applied into graph theory. The conjugacy class graph of G is proven as a complete graph. Consequently, some properties of the graph related to conjugacy classes of the group are found. This includes the number of connected components, diameter, the number of edges and the regularity of the graph. Furthermore, the clique number and chromatic number for groups of Type 1, 2 and 3 are shown to be identical. Besides, some properties of the graph related to commuting conjugacy classes of abelian and dihedral groups are introduced.

ABSTRAK

Dua unsur a dan b bagi suatu kumpulan G disebut sebagai konjugat sekiranya wujud suatu unsur g dalam G yang mana $gag^{-1} = b$. Set bagi semua konjugat dalam suatu kumpulan G membentuk kelas kekonjugatan. Objektif utama penyelidikan ini adalah untuk menentukan bilangan kelas kekonjugatan dan saiz kelas kekonjugatan bagi kumpulan-2 berpenjana-2 dengan kelas nilpoten dua. Andaikan G ialah suatu kumpulan-2 berpenjana-2 dengan kelas dua yang mana merangkumi tiga jenis iaitu Jenis 1, Jenis 2 dan Jenis 3. Rumus am untuk bilangan kelas kekonjugatan bagi G ditentukan dengan menggunakan kumpulan asas dan kaedah perlanjutan pusat. Bagi setiap jenis dalam kumpulan G, bilangan kelas kekonjugatannya telah ditunjukkan terdiri daripada dua rumus am. Tambahan pula, saiz kelas kekonjugatan dikira berdasarkan peringkat bagi subkumpulan terbitan. Gambaran yang mendalam diperoleh dengan mengaplikasikan setiap keputusan ke dalam teori graf. Graf bagi kumpulan G dibuktikan sebagai satu graf lengkap. Akibatnya, beberapa ciri bagi graf yang berkaitan dengan kelas kekonjugatan bagi kumpulan G ditemui. Ciri ini merangkumi bilangan komponen terkait, diameter, bilangan tepi dan kenalaran bagi graf G. Tambahan pula, nombor klik dan nombor berkroma dibuktikan sama bagi kumpulan Jenis 1, 2 dan 3. Selain itu, beberapa ciri bagi graf yang berkaitan dengan kelas kekonjugatan yang berkalisan tukar tertib bagi kumpulan abelan dan kumpulan dwisatah telah diperkenalkan.

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LIST OF SYMBOLS

1	–	Identity element
$\langle a \rangle$	–	Cyclic subgroup generated by a
C_n	–	Cyclic group of order n
$C(a)$	–	Centralizer of a in G
$cl(a)$	–	Conjugacy class of a
cl_G	–	Number of conjugacy classes
D_n	–	Dihedral group of order $2n$
$d(\Gamma_G)$	–	Diameter of a graph G
$d(x, y)$	–	Distance between x and y in a graph G
$E(\Gamma_G)$	–	Edge set of a graph G
$\|G\|$	–	Order of the group G
G'	–	Commutator subgroup of G
$\|G : H\|$	–	Index of the subgroup H in the group G
G/H	–	Factor group
$G \times H$	–	Direct product of G and H
$G \rtimes H$	–	Semidirect product of G and H
$G \cong H$	–	G is isomorphic to H
$H \leq G$	–	H is a subgroup of G
$H \triangleleft G$	–	H is a normal subgroup of G
Ker α	–	Kernel of the homomorphism α
$\mathbb{N}$	–	Set of natural numbers
$n(\Gamma_G)$	–	Number of connected components of G
$P(G)$	–	Commutativity degree
Q_n	–	Quaternion group of order $2n$
$V(\Gamma_G)$	–	Vertex set of a graph G
$\omega(\Gamma_G)$	–	Clique number of a graph G
$\|x\|$	–	Order of the element x

$[x, y]$	–	The commutator of x and y
$\langle X \mid R \rangle$	–	Groups presented by generators X and relators R
$\chi(\Gamma_G)$	–	Chromatic number of a graph G
$\mathbb{Z}$	–	Set of integers
$\mathbb{Z}/n\mathbb{Z}$	–	Integers modulo n
$Z(G)$	–	Center of the group G
Γ_G	–	Graph related to conjugacy classes of a group G
γ_G	–	Graph related to commuting conjugacy classes of a group G

CHAPTER 1

INTRODUCTION

1.1 Introduction

In group theory, a fruitful method of partitioning elements from all range of groups into classes is through conjugacy classes. Thus, each element in a group belongs to exactly one class. Elements from the same class will share their properties. A classical problem in group theory is to investigate the number of conjugacy classes, conjugacy class sizes and the graph related to it.

Suppose that a and b are elements of a group G, then we say that a and b are conjugate in G if $xax^{-1} = b$ for some $x \in G$. Let cl_G denotes the number of conjugacy classes of a group G. Conjugation is an equivalence relation on G. This equivalence relation induces a partition of G whose elements are called conjugacy classes. Thus, any group may be partitioned into distinct conjugacy classes. Note that by the properties of equivalence relation, the union of all distinct conjugacy classes of a group is the entire group itself.

The set $cl(a) = \{xax^{-1} \,|x \in G\}$ denotes the conjugacy class of a. Two conjugacy classes, $cl(a)$ and $cl(b)$, are equal if and only if a and b are conjugate. The identity element is always in its own conjugacy class, that is, $cl(1) = \{1\}$. The conjugacy class of an abelian group is a singleton since each element is in its own conjugacy classes. Other than that, for the trivial group, the number of conjugacy classes is exactly one. Besides, for the non-trivial group, $cl_G > 1$.

In this research, the number of conjugacy classes and conjugacy class sizes of 2-generator 2-groups of class two are determined. The results are then applied to graph theory and some properties of graph related to conjugacy classes of 2-generator 2-groups of class two are found.

1.2 Research Background

The concepts of conjugacy classes have sparked many interests among researchers in group theory over the past decades. The estimation of the number of conjugacy classes for finite groups has already been considered by many authors [1–5] since Landau's observation in 1903. Later in 1979, Sherman [6] improved the lower bound for nilpotent group, but yet not tight. After 1979, many other researchers tried improving this bound. In 2008, Ahmad [7] computed the exact number of conjugacy classes for 2-generator p-groups of nilpotency class two where p is an odd prime, while in this research, the exact number of conjugacy classes of 2-generator 2-groups of class two are determined. The difference between Ahmad work for p-groups and 2-groups is mainly the classification used. The classification by Kappe *et al.* [8] which is later modified by Magidin [9] is used in this study.

Besides, the studies on the influence of the size of the conjugacy classes on the structure of a finite group have been the subject of research over the years. Many researchers produced papers on this topic, for instance [10–17]. However, very little is known about how the conjugacy class sizes depend on the order of the commutator subgroup.

A deeper insight is gained by applying the conjugacy classes and conjugacy class sizes into graph theory. There has been considerable work over the years on the graph related to conjugacy classes, Γ_G, where G denotes a finite group or an infinite FC-group. Earlier research on the graph of conjugacy classes was introduced by Bertram *et al.* in [18]. The properties of the graphs associated to conjugacy classes of groups have already been considered by many authors [18–22]. This

research focuses on some properties including the number of connected components, diameter, the number of edges and the regularity of the graph of 2-generator 2-groups of class two. Besides, the clique number and chromatic number for these groups are also determined.

However, less research has been done on the graphs related to commuting conjugacy classes. In 2009, the possible structure of a periodic solvable group and locally finite group are investigated in [23], which include, connectivity and diameter. Hence, some properties of graphs related to commuting conjugacy classes are explored in this research.

1.3 Problem Statements

Conjugacy classes have already been there since the turn of the century. However, for many years there have been only lower and upper bounds for the number of conjugacy classes. In 2008, Ahmad [7] found the general formula for the exact number of conjugacy classes for 2-generator p-groups of class two (p an odd prime). Moreover, there is only one paper produced on graph of commuting conjugacy classes in 2009. The possible structure of a periodic solvable group and locally finite group are investigated in [23]. Hence, in this research, the following questions will be addressed and answered.

(i) What is the exact number of conjugacy classes of 2-generator 2-groups of class two?

(ii) What is the conjugacy class sizes of 2-generator 2-groups of class two?

(iii) Can the conjugacy class sizes be computed based on the order of the derived subgroup?

(iv) What is the application of conjugacy classes and conjugacy class sizes of 2-generator 2-groups of class two?

(v) What are the properties for the graph related to conjugacy classes of 2-generator 2-groups of class two?

(vi) What are the properties for the graph of commuting conjugacy classes for abelian and dihedral groups?

1.4 Research Objectives

The objectives of this research are to:

(i) develop general formulas for the number of conjugacy classes of 2-generator 2-groups of class two,

(ii) obtain the conjugacy class sizes of 2-generator 2-groups of class two,

(iii) determine the number of connected components, diameter, clique number, chromatic number, the number of edges and the regularity of conjugacy class graph for 2-generator 2-groups of class two,

(iv) introduce the completeness, number of connected components, diameter, chromatic number and the probability of commuting conjugacy class graph.

1.5 Scope of the Study

In this thesis, only 2-generator 2-groups of class two are considered in computing the number of conjugacy classes, conjugacy class sizes and graph related to conjugacy classes including the number of connected components, diameter, the number of edges, the regularity of graph, clique number and chromatic number. The classification of 2-generator 2-groups of class two by Kappe *et al.* [8] which is later modified by Magidin [9] is used in this study.

1.6 Significance of Findings

The results on the conjugacy classes can be used in other applications such as in graph theory. In compliance with the earlier topic on the conjugacy classes

and conjugacy class sizes, a graph can be introduced. Vertex set and two distinct vertices can be connected with an edge. It is worth pointing out that many results can be interpreted in the language of graph theory. There are many interesting problems for instance, the number of connected components, diameter, regularity, connections between the structure of the graph and the structure of the group.

In probabilistic group theory, the degree of abelianness of a group can be computed using its conjugacy classes. The degree of abelianness of a group, or sometimes called the commutativity degree, $P(G)$ is defined as $P(G) = \frac{|\{(x,y)\in G\times G|xy=yx\}|}{|G|^2}$. Gustafson in [4] showed that $P(G) = \frac{cl_G}{|G|}$, where cl_G is the number of conjugacy classes of a group G.

1.7 Research Methodology

This research begins by studying the classification of the 2-generator p-groups of class two by Bacon and Kappe [24]. Kappe *et al.* [8] classified the 2-generator 2-groups of class two and Magidin [9] modified the classification in terms of generator and relations. Based on these classifications, determination of the number of conjugacy classes of 2-generator p-groups of class two (p is an odd prime) was done by Ahmad in [7]. Furthermore, the methods used by Ahmad was studied. Firstly, the base group and the central extension of these 2-groups are computed. Secondly, the order of the group and order of the center of the 2-groups are formulated. By these computations, some lemmas and theorems are developed. Then, the conjugacy class sizes of these groups are determined. These results are then applied into graph theory. The number and the size of conjugacy classes is used to determine the vertices of the conjugacy class graph and commuting conjugacy class graph, respectively. Then some properties of the graph related to conjugacy classes of some 2-groups are found. They are the number of connected components, diameter, the number of edges and the regularity of the graph. Besides, some properties of the commuting conjugacy class graph of abelian groups and dihedral groups are found. Groups, Algorithms and Programming (GAP) software has been

used to help in facilitating the computations and give us the general idea of the number and size of conjugacy classes.

1.8 Thesis Organization

This thesis is divided into seven chapters. The first chapter serves as an introduction to the whole thesis, including research background, problem statement, research objectives, scope, significance of findings and research methodology.

Chapter 2 presents the literature review of this research. Some basic definitions and concepts related to this research are presented. Various works by different researchers concerning the bound and the exact number of conjugacy classes, conjugacy class sizes and graphs related to conjugacy classes are compared and stated. Furthermore, the classifications of 2-generator 2-groups of class two and the application of GAP in this research are given in this chapter.

Chapter 3 focuses on the usage of Groups, Algorithms and Programming (GAP) software to construct the number of conjugacy classes and the conjugacy class sizes of 2-generator 2-groups of class two. Moreover, some other properties of these groups are also computed using GAP. Some examples using GAP commands are specified and illustrated (please refer to Appendix A).

In Chapter 4, some preliminary results on the number of conjugacy classes are stated with the proofs. In addition, the main results of the computations of the exact number of conjugacy classes of 2-generator 2-groups of class two are given according to their types and aided with some examples.

The second main result of this thesis is given in Chapter 5. The conjugacy class sizes of 2-generator 2-groups of class two are stated with the proofs.

In Chapter 6, the results from Chapter 4 and 5 are used and applied in graph theory. The simple graphs of 2-generator 2-groups of class two are defined.

Some properties of these graphs including the number of connected components, diameter and the regularity of these graphs are determined. Besides, the general formula for the number of edges of 2-generator 2-groups of class two are also given depending on their types. Furthermore, the connections between the clique number and the chromatic number of these groups are shown too.

Finally, the summarization of the whole thesis and suggestions for future research are concluded in Chapter 7. Moreover, this chapter also introduces the commuting conjugacy class graph of a group. Some properties which are enacted to this graph are given for abelian groups and dihedral groups. These include the completeness, connectivity, diameter, chromatic number, clique number and the probability of commuting conjugacy classes.

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