NUMERICAL METHODS FOR NONLINEAR SYSTEMS OF EQUATIONS

WONG EE CHYN

A thesis submitted in fulfillment of the requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
University Technology of Malaysia

JAN 2014

To my beloved parents, brother,sisters and all my friends,

Thanks for all your love and support.

ACKNOWLEDGEMENT

First of all, I would like to dedicate my greatest appreciation to my project supervisor, PM Dr. Munira Ismail for her guidance throughout this research. She spends a lot of time to guide me and help me in processing my thesis. My gratitude to her, her dedication has given me a learning experience in writing this thesis.

Secondly, I would also like to dedicate my appreciation to Dr. Yeak Su Hoe who had taken a lot of effort to give helpful suggestions during this study.

Moreover, I would also like to express my gratitude to my family members, especially my father and mother in encouraging me in doing this project. Their love and advice have always supported me throughout this research.

Last but not least, I would also like to thank those who have helped me directly or indirectly to complete my research, mainly my friends and course mates. They always support me by providing some information and idea and showed interest in the progress of my thesis.

Once again, I would like to convey my deepest appreciation to all of you. Thanks for your kindness and tolerance all the way.

ABSTRACT

It is common to have nonlinear systems of equations to be solved in numerical application. However, such nonlinear systems of equations are difficult to be solved either exactly or numerically. There are several methods that can be used to solve the nonlinear systems of equations numerically such as Newton's method, quasi-Newton method, and homotopy continuation method. Some numerical examples of nonlinear systems of equations are shown in this study. Further, a heat transfer process is model as a problem that nonlinear system of equations is solved with the methods that had been mentioned earlier. The numerical results are computed by using MATLAB codes and the results are compared in order to determine the accuracy of these three methods.

ABSTRAK

Sistem persamaan tak linear sering terlibat dalam aplikasi berangka. Walau bagaimanapun, sistem persamaan tak linear ini susah untuk diselesaikan sama ada dengan kaedah tepat atau kaedah berangka. Terdapat beberapa kaedah yang boleh digunakan dalam menyelesaikan masalah sistem persamaan tak linear. Sebagai contoh, kaedah Newton, kaedah kuasi-Newton, dan kaedah kesinambungan homotopi yang melibatkan sistem persamaan tak linear telah ditunjukkan dalam kajian ini. Di samping itu, proses pemindahan haba adalah sebagai satu masalah yang melibatkan penyelesaian sistem persamaan tak linear dengan menggunakan keadah-kaedah yang disebut sebelum ini. Semua pengiraan adalah dilakukan melalui MATLAB komputer kod. Hasil pengiraan yang didapati telah dibandingkan untuk mengetahui tahap ketepatan antara tiga kaedah yang digunakan.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

$\mathbb{R}^n$	-	Set of ordered n-tuples of real numbers
$\boldsymbol{x}$	-	Column vector or element of $\mathbb{R}^n$
∇g	-	Gradient of the function g
$\mathbf{0}$	-	Vector with all zero entries
$O(.)$	-	Order of convergence
$\rightarrow$	-	Equation replacement
δ	-	Delta
A^{-1}	-	Inverse matrix of the matrix A
A^t	-	Transpose matrix of the matrix A
$\lVert\boldsymbol{x}\rVert$	-	Arbitrary norm of the vector **x**
$\|\|\boldsymbol{x}\|\|_2$	-	The l_2 norm of the vector **x**
$\lVert\boldsymbol{x}\rVert_\infty$	-	The l_∞ norm of the vector **x**
$A(\boldsymbol{x})$	-	Matrix whose entries are the functions form $\mathbb{R}^n$ into $\mathbb{R}$
$J(\boldsymbol{x})$	-	Jacobian matrix
J_h	-	Radiosities of the heater
J_c	-	Radiosities of coating surfaces
T_h	-	Temperatures of the heater
T_c		Temperatures of coating surfaces
N		Number of subintervals

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of Problem

In all areas of science and engineering, equations need to be solved. An equation of one variable can be written as

$$f(x) = 0. \tag{1.1}$$

A numerical value of x that satisfies the equation is a solution to the equation or called the root of the equation. According to Gilat and Subramaniam (2008), the value of x can be determined analytically when the equation is simple. However, it is impossible to determine the root of an equation analytically in many situations. A numerical solution of an equation is a value of x that satisfies the equation approximately. This means than the value of $f(x)$ is close to zero but not exactly zero when the value of x is substituted into the equation.

Nonlinear system in mathematics is a system that does not satisfy the superposition principle or the output of the system is not directly proportional to its input. The equation in any problem cannot be written as a linear combination of unknown variables or functions for a nonlinear system.

A linear equation in $x, y, z, \ldots$ can be written in the form

$$ax + by + cz + \cdots = constant \tag{1.2}$$

with $a, b, c, \ldots$ being constant.

Below is shown the examples of linear equation.

$$2x - 6y + z = 3 \tag{1.3}$$

$$4s + t - u = -1 \tag{1.4}$$

The equations (1.3) and (1.4) are the linear equations.

An equation that consists of expressions such as

$$xy, x^2, y^{-2}, (2z - y)^2, \frac{\sqrt{x}}{y}, \cos y\,, e^{xz}, z\sqrt{x + y}$$

is known as nonlinear because it cannot write as a linear equation (1.2). Nonlinear equations are more difficult to be solved if compare with linear equations even the number of unknowns is small (Linz and Wang, 2003).

A nonlinear equation is a function f such that

$$f(x) = 0 \tag{1.5}$$

and the value of x for which f is zero is the root of equation, or zero of function f. This problem is known as root finding. A nonlinear equation can be solved numerically by using Newton's method, secant method, and fixed- point iteration method.

Examples of nonlinear equation are shown at below.

$$x^2 - 4sinx = 0 \tag{1.6}$$

$$3x^2 + sinx - e^x = 0 \tag{1.7}$$

A system of n equations and n unknowns $x_1, x_2, \dots, x_n$ is nonlinear if exists of one or more nonlinear equations. With n unknowns, $x_1, x_2, \dots, x_n$, a system of n simultaneous nonlinear equations has the form:

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0. \end{aligned} \tag{1.8}$$

Following is showing an example of a nonlinear system of equations.

$$\begin{aligned} f_1(x_1, x_2, x_3) &= 3x_1 - cos(x_2 x_3) - \tfrac{1}{2} = 0 \\ f_2(x_1, x_2, x_3) &= x_1^2 - 81(x_2 + 0.1)^2 + sin x_3 + 1.06 = 0 \\ f_3(x_1, x_2, x_3) &= e^{-x_1 x_2} + 20x_3 + \tfrac{10\pi - 3}{3} = 0 \end{aligned} \tag{1.9}$$

Nonlinear systems of equations appear in numerical applications frequently. The nonlinear systems of equations are usually difficult to solve, either exactly or numerically (Scheffel and Hakansson, 2009). Several methods can be used to solve a nonlinear system of equations numerically, such as Newton's methods, quasi-Newton methods, steepest descent techniques, and homotopy continuation methods.

1.2 Problem of Statement

Recently, nonlinear systems of equations have occurred in many important fields such as engineering, mechanics, medicine, chemistry, and robotics. The aim of this study is to solve nonlinear systems of equations using homotopy continuation method and compare with Newton's method and quasi-Newton method to see the accuracy of the methods. After that, apply the nonlinear system of equations in the heat transfer processes for a coating on the panel surface, and then use these three methods to get the approximation solutions.

1.3 Objective of The Study

The objectives of this research are:

i. To solve nonlinear system of equations by using Newton's method, quasi-Newton method, and homotopy continuation method, and then compare the results of these methods.

ii. To write computer codes of Newton's method, quasi-Newton method, and homotopy continuation method for solving nonlinear systems of equations by using MATLAB.

iii. To apply the nonlinear systems of equations in the heat transfer processes for a coating on the panel surface and solved by using Newton's method, quasi-Newton method, and homotopy continuation method.

1.4 Scope of the Study

In this study, we will solve the nonlinear systems of equations by using several methods. Nonlinear system of equations can be solved by using Newton's methods, quasi-Newton methods, fixed-point iteration, steepest descent techniques, homotopy continuation methods, Levenberg-Marquardt methods and others. We have chosen three methods that can be used to solve the nonlinear systems. The first method is Newton's method which is the most common method for solving the nonlinear systems of equations. Then, quasi-Newton method which is modified from the Newton's method will also be used to solve the nonlinear system of equations. The last method that will use in this study is the homotopy continuation method.

There are many problems where nonlinear system of equations occurs such as engineering, mechanics, medicine, chemistry, and robotics. One application is presented in this work which is the heat transfer processes for a coating on the panel surface. This problem can be represented by a nonlinear system of equations and here the problem will be solved with these three methods. In addition, computer codes of these three methods for solving the nonlinear system of equations will be written by using MATLAB to get the numerical solutions.

1.5 Significance of the Study

Recently, nonlinear systems of equations appear in numerical applications frequently. Normally, we will use Newton's method and quasi-Newton method to solve the nonlinear systems of equations. In this study, we employ another method, homotopy continuation method to use in solving nonlinear systems of equations. So, this study will illustrate the performance of the three methods for solving nonlinear systems of equations.

In this study, we will understand more about the nonlinear system of equations that can be solved using the Newton's method, quasi-Newton method and homotopy continuation method.

REFERENCES

Allgower, E., and Georg, K. (1980). Simplicial and continuation methods for approximating fixed points and solutions to systems of equations. *SIAM Rev*. , 22 (1), 28.

Aslam Noor, M., and Waseem, M. (2009). Some iterative methods for solving a system of nonlinear equations. *Computers and Mathematics with Applications* , 57, 101-106.

Bader, B. W. (2005). Tensor-Krylov methods for solving large-scale systems of nonlinear equations. *SIAM J. Numerical Analysis* , 43 (3), 1321-1347.

Brown, K. M. (1969). A quadratically convergent Newton-like method based on upon Gassian elimination. *SIAM Journal on Numerical Analysis* 6 , 560-569.

Broyden, C. G. (1965). A class of methods for solving nonlinear simultaneous equations. *Macthematics of Computation* 19 , 577-593.

Burden, R. L., and Faires, J. D. (2011). *Numerical Analysis* (Ninth Edition ed.). Canada: Nelson Education, Ltd.

Chow, J. *et al*. (1991). Homotopy Continuation Methods for Neural Networks. *ISCAS Conference* (pp. 2483-2486). Singapore.

Dennis, J. E., and More, J. J. (1977). Quasi-Newton methods, motivation and theory. *SIAM Review* 19 , 46-89.

Effati, S., and Nazemi, A. R. (2005). A new method for solving a system of the nonlinear equations. *Appl. Math. Comput*. , 168 (2), 877-894.

Fausett, L. V. (2003). *Numerical Methods : Algorithms and Applications*. United States of America: Pearson Education, Inc. .

Frontini, M., and Sormani, E. (2004). Third-order methods from quadrature formulae for solving systems of nonlinear equations. *Applied Mathematics and Computation* , 149, 771-782.

Fukushima, M. (1988). Solving inequality constrained optimization problems by differential homotopy continuation methods. *Journal of Mathematical Analysis and Applications* , 133, 109-121.

Gilat, A., and Subramaniam, V. (2008). *Numerical Method for Engineers and Scientists*. United State of America: John Wiley and Sons, Inc. .

Gonzalez-Lima, M. D., and Montes de Oca, F. (2009). A Newton-like method for nonlinear system of equations. *Numer. Algor*. , 52, 479-506.

Grosan, C., and Abraham, A. (2008). A New Approach for Solving Nonlinear Equations Systems. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans* , 38 (3), 698-714.

Hirsch, M. J. *et al.* (2007). Global optimization by continuous grasp. *Opt. Lett*. , 1 (2), 201-212.

Kalaba, R., and Tesfatsion, L. (1991). Solving nonlinear equations by adaptive homotopy continuation. *Applied Mathematics and Computation* , 41, 99-115.

Kou, J. S. *et al.* (2006). Efficient continuation Newton-like method for solving systems of nonlinear equations. *Applied Mathematics and Computation* , 174, 846-853.

Kuroe, Y., and Kato, T. (1996). A computer method for finding operating-mode boundaries in power electronic circuits by curve tracing. *IEEE* , 66-70.

Linz, P., and Wang, R. (2003). *Exploring Numerical Methods: An Introduction to Scientific Computing using MATLAB.* United States of America: Jones and Bartlett Publishers, Inc. .

Liu, H., and Ni, Q. (2008). Incomplete Jacobian Newton method for nonlinear equations. *Comp. and Math. with Appl*. , 56, 218-227.

Martinez, J. M. (2000). Practical quasi-Newton methods for solving nonlinear systems. *Journal of Computational and Applied Mathematics* , 124, 97-121.

More, J., and Cosnard, M. Y. (1979). Numerical solution of nonlinear equations . *ACM Transactions on Mathematical Software* 5 , 64-85.

Morgan, A. (2009). *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems.* New Jersey: Prentice-Hall.

Ortega, J. M., and Rheinbolt, W. C. (1970). *Iterative Solution of Nonlinear Equations in Several Variables.* New York: Academic.

Rao, M. A., and Srinivas, J. (2003). *Neural Networks Algorithms and Applications*. India: Alpha Science International Ltd.

Rion, W. L., and Van Brunt, V. (1990). Differential geometry based homotopy continuation. *Computer Chem. Engng.* , 14 (8), 889-900.

Samarasinghe, S. (2007). *Neural Networks for Applied Sciences And Engineering*. United State of America: Taylor & Francis Group, LLC.

Scheffel, J., and Hakansson, C. (2009). Solution of systems of nonlinear equations - a semi-implicit approach. *Applied Numerical Mathematics* , 59, 2430-2443.

Tari, H. *et al.* (2010). A constrained homotopy technique for excluding unwanted solutions form polynomial equations arising in kinematics problems. *Mechanism and Machine Theory* , 45, 898-910.

Wang, X. (1994). An Efficient Training Algorithm for Multilayer Neural Networks by Homotopy Continuation Method. *IEEE* 494-497.

Wayburn, T. L., and Seader, J. D. (1987). Homotopy continuation methods for computer-sided process design. *Comput. chem. Engng.* , 11, 7-25.

Wu, T. M. (2006). The inverse kinematics problem of spatial 4P3R robot manipulator by the homotopy continuation method with an adjustable auxiliary homotopy function. *Nonlinear Analysis* , 64, 2373-2380.

Wu, X. (2007). Note on the improment of Newton's method for system of nonlinear equations. *Appl. Math. Comput.* , 189, 1476-1479.

Wu, X., and Fu, D. (2001). New high-order convergence iteration methods without employing derivatives for solving non-linear equations. *Comput. Math. Appl.* , 41, 489-495.

Wu, X., and Wu, H. (2000). On a class of quadratic convergence iteration formulae without derivatives. *Appl. Math. Comput* , 107, 77-80.

Zhang, L. Q., and Han, G. Q. (1993). Structure variable homotopy methods for solving nonlinear systems. *Applied Mathematics and Computation* , 59, 177-191.

Zhu, D. *et al.* (2004). A homotopy method for solving bilevel programming problem. *Nonlinear Analysis* , 57, 917-928.