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Numerical Computation of a Two-Dimensional Navier-Stokes Equation Using an Improved Finite Difference Method

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Abstract The two-dimensional Navier-Stokes equation is solved numerically using a finite differenced based method which essentially takes advantage of the best features of two well-established numerical formulations, the finite difference and the finite volume methods. The lid-driven cavity flow is validated using an improved finite difference method. It is found that within the family of the finite difference methods used for the solution of steady and incompressible flows, this new approach provides a viable alternative for handling the pressure of the flow.

Keywords finite difference method; Navier-Stokes equations; incompressible flow; staggered grid; pressure correction equation

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1 Introduction

Numerical simulation of fluid flow has been a major topic of research for the past few decades, [1]. Computational fluid dynamics (CFD) is one of the prominent physical disciplines that involves the description of the fluid flow in terms of mathematical models that include convective and diffusive transport of some variables. These mathematical models consist of a set of governing equations in the form of ordinary or partial differential equations. Over the years, the finite difference method (FDM) is frequently used in CFD.

One of the main challenges in the FDM is in the handling of the pressure of the flow. In general, no physical specification of pressure exists. Even though there are three equations for the three unknowns u, v, p, there is no explicit equation which can be used for pressure. In most finite difference solution schemes for incompressible steady flows, the pressure field is obtained from a Poisson equation which is derived from the momentum equations and the continuity equation, [2]. The difficulty inherent in this approach is the need to decide on additional boundary conditions for the pressure [3]. Petersson [4] discussed this problem in details.

To overcome this problem, Zhang et al. [5] used a fully explicit second-order accurate time marching scheme via a pointbased compact finite difference method,where the pressure Poisson equation was solved by a pseudo-time marching procedure. Petersson [4] proposed a new scheme that was implemented with SIMPLE-type algorithm for the pressure field calculation similar to that of finite volume methods. The discretised equations were developed as a purely finite difference formulation. The convective terms in the momentum equations were approximated using the first or second order finite difference formula.

A number of experimental and numerical studies have been conducted to investigate the flow field of a lid-driven cavity flow in the last several decades ([1], [7], [8], [9]). It is known that the lid-driven cavity flow problem is not only technically important but also of great scientific interest because it displays almost all fluid mechanical phenomena in the simplest geometrical settings, [10]. The problem is also attractive because of its importance in industrial applications such as coating and drying technologies, melt spinning processes and many others, [11]. The present work is concerned with the validation of a lid-driven cavity flow using a new FDM approach.

2 Governing Equations

The dimensionalised governing equations of the fluid flow are given respectively by the continuity equation

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

x-momentum equation

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right) \tag{2}$$

y-momentum equation

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right) \tag{3}$$

where u and v are the velocity components in the x and y directions respectively, p is the pressure, ρ is the constant density, and ν is the viscosity.

Using the dimensionless definitions, [12],

$$t = \frac{t^* U}{h}, x = \frac{x^*}{h}, y = \frac{y^*}{h}, u = \frac{u^*}{U}, v = \frac{v^*}{U}, p = \frac{p^*}{\rho U^2}.$$

the governing equations (1) to (3) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{5}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{6}$$

where $Re = Uh/\nu$ is the Reynolds number.

3 Numerical Method

3.1 Finite Differencing on a Staggered Grid

Consider a two-dimensional rectangular cavity flow domain which is discretised using a regular Cartesian mesh as shown in Figure 1. The mesh is uniform in x and y directions. A staggered grid is used to store the velocity components u and v and the pressure p.

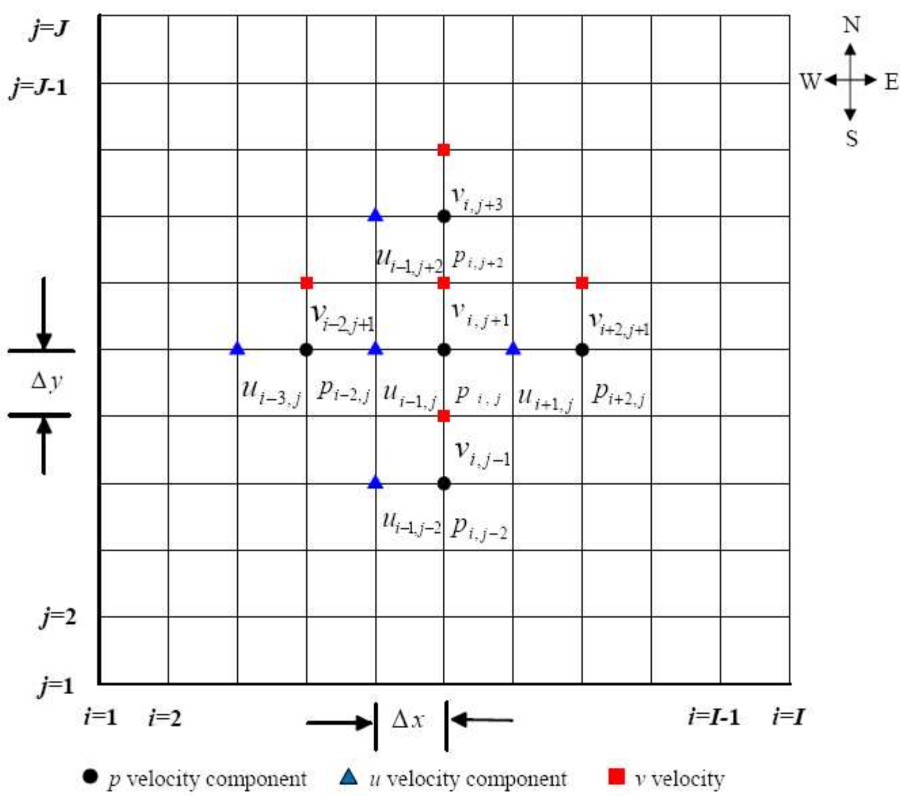

Figure 1: Staggered Grid Arrangement

As indicated in Figure 1, the values of u and v are stored at the i-1,j and i,j+1 locations respectively and p is stored at i,j. Thus, the u-momentum (equation 5) is discretised at i-1,j, the v-momentum (equation 6) at i,j+1, and the continuity (equation 4) at i,j. Here, a first-order upwind differencing scheme is used to approximate the convective terms in the momentum equations, while a second-order central differencing is used for the diffusion terms. The pressure gradients are approximated by a second order central difference scheme.

3.2 Discretisation of the Momentum Equations

Zogheib [6] used unequally spaced grid points for handling the u- and v-momentum equations at the wall boundary, i.e. the convective term is approximated using a second order accurate expression while the diffusion term takes the first order accurate expression. It hence leads to the creation of different formulae for different node locations. On the other hand, equally spaced grid points are chosen in this study. Discretisation of the u- and v-momentum equations at interior nodes can be used at the wall boundary. The discretisation of the momentum equations are summarized as follows.

3.2.1 The u-Momentum Equation

The discrete u-momentum equations at the interior nodes are given by

$$a_P^{int}u_{i-1,j} + a_N^{int}u_{i-1,j+2} + a_S^{int}u_{i-1,j-2} + a_W^{int}u_{i-3,j} + a_E^{int}u_{i+1,j} = \frac{\hat{p}_{i-2,j} - \hat{p}_{i,j}}{2\rho\Delta x}$$

where

$$\begin{aligned}
a_P^{int} &= \frac{\hat{u}_{i-1,j}}{2\Delta x} + \nu\left(\frac{1}{2\Delta x^2} + \frac{1}{2\Delta y^2}\right)\\
a_N^{int} &= \frac{\hat{v}_{i-1,j}}{4\Delta y} - \frac{\nu}{4\Delta y^2}\\
a_S^{int} &= -\frac{\hat{v}_{i-1,j}}{4\Delta y} - \frac{\nu}{4\Delta y^2}\\
a_W^{int} &= -\frac{\hat{u}_{i-1,j}}{2\Delta x} - \frac{\nu}{4\Delta x^2}\\
a_E^{int} &= -\frac{\nu}{4\Delta x^2}
\end{aligned}$$

Variables with the carets above them are the quantities that will be calculated at the previous iteration. Because of the use of a staggered grid, the values of v in the u-momentum equation and u in the v-momentum equation, appearing as the coefficients of the convective derivatives, are not available at the desired points.

Therefore, these velocities are computed to a second order accuracy using the velocities of four surrounding grid points at which they are stored,

$$u|_{i,j+1} = \frac{u_{i+1,j} + u_{i+1,j+2} + u_{i-1,j} + u_{i-1,j+2}}{4}$$

$$v|_{i-1,j} = \frac{v_{i,j-1} + v_{i,j+1} + v_{i-2,j-1} + v_{i-2,j+1}}{4}$$

The discrete u-momentum equations at boundary nodes are the same as interior node with some modifications. For example, the discrete u-momentum equations at the inlet nodes is the same as interior node except that the value of $u_{1,j}$ is known.

3.2.2 The v-Momentum Equation

Analogous to the discrete u-momentum equations, the discrete v-momentum equations at the interior nodes take the following form

$$b_P^{int}v_{i,j+1} + b_N^{int}v_{i,j+3} + b_S^{int}v_{i,j-1} + b_W^{int}v_{i-2,j+1} + b_E^{int}v_{i+2,j+1} = \frac{\hat{p}_{i,j} - \hat{p}_{i,j+2}}{2\rho\Delta y}$$

where

$$
\begin{aligned}
b_P^{int} &= \frac{\hat{u}_{i,j+1}}{2\Delta x} + \nu\left(\frac{1}{2\Delta x^2} + \frac{1}{2\Delta y^2}\right) \\
b_N^{int} &= \frac{\hat{v}_{i,j+1}}{4\Delta y} - \frac{\nu}{4\Delta y^2} \\
b_S^{int} &= -\frac{\hat{v}_{i,j+1}}{4\Delta y} - \frac{\nu}{4\Delta y^2} \\
b_W^{int} &= -\frac{\hat{u}_{i,j+1}}{2\Delta x} - \frac{\nu}{4\Delta x^2} \\
b_E^{int} &= -\frac{\nu}{4\Delta x^2}
\end{aligned}
$$

3.3 Discretisation of the Continuity Equation

The pressure correction equations for all boundary nodes are identical to those given in [6]. The pressure correction equations for the interior nodes are

$$
c_P^{int} p_{i,j}^{'} + c_E^{int} p_{i+2,j}^{'} + c_W^{int} p_{i-2,j}^{'} + c_N^{int} p_{i,j+2}^{'} + c_S^{int} p_{i,j-2}^{'} = \frac{u_{i-1,j}^{*} - u_{i+1,j}^{*}}{2\Delta x} - \frac{v_{i,j+1}^{*} - v_{i,j-1}^{*}}{2\Delta y}
$$

where

$$
\begin{aligned}
c_P^{int} &= \frac{1}{4\rho\Delta x^2 a_{i+1,j}} + \frac{1}{4\rho\Delta x^2 a_{i-1,j}} + \frac{1}{4\rho\Delta y^2 b_{i,j+1}} + \frac{1}{4\rho\Delta y^2 b_{i,j-1}} \\
c_E^{int} &= -\frac{1}{4\rho\Delta x^2 a_{i+1,j}} \\
c_W^{int} &= -\frac{1}{4\rho\Delta x^2 a_{i-1,j}} \\
c_N^{int} &= -\frac{1}{4\rho\Delta y^2 b_{i,j+1}} \\
c_S^{int} &= -\frac{1}{4\rho\Delta y^2 b_{i,j-1}}
\end{aligned}
$$

3.4 Solution Algorithm

It is customary to use the SIMPLE scheme for the pressure-velocity coupling in the overall solution. The procedure can be summarised in the following steps:

i) Start with an initial pressure field p^*.

ii) Solve the momentum equations to obtain the intermediate velocity components U^*, V^*.

iii) Solve the pressure-correction equation to obtain the pressure correction $p^{'}$.

iv) Update the intermediate pressure and velocity fields with the corrected values $p = p^* + p^{'}$, $U = U^* + U^{'}$, $V = V^* + V^{'}$.

v) Take the newly corrected pressure p as a new initial pressure p^* and repeat the entire procedure until a fully converged solution is obtained.

Note that the pressure-correction equation is also prone to divergence unless an under-relaxation is implemented.

Zogheib [6] used the Tri-Diagonal Matrix Algorithm (TDMA) to solve the system of discretised equations. However, this algorithm takes a considerable long time to converge. In the present algorithm, a modification is provided in the form of a direct method to remedy the problem. It has been shown that a rapid convergence can be achieved through this method, [13].

4 Numerical Results

The developed finite difference formulation is applied to a well-establised benchmark problem, namely, the flow in a lid-driven cavity. This problem is considered to be important validation test case for any numerical method. The lid-driven cavity problem has long been used as the test or validation case for new codes or new solution methods. The problem geometry is simple and two-dimensional, and the boundary conditions are also simple. The standard case is fluid contained in a square domain with the Dirichlet boundary conditions on all sides, with three stationary sides and one moving side (with a prescribed velocity tangent to the side). The boundary conditions for the present problem are illustrated in Figure 2.

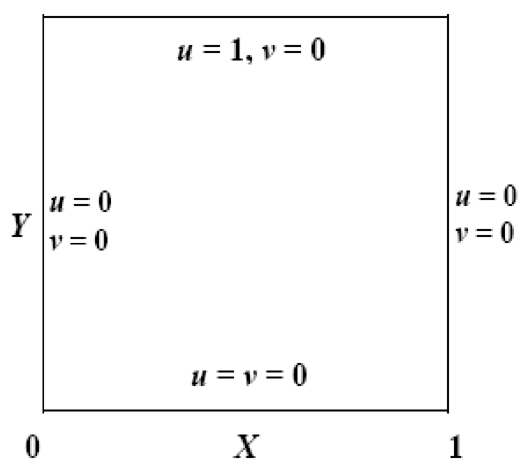

Figure 2: Boundary Conditions for the Lid-driven Cavity

A good set of data for comparison is the data of Ghia at el. [1]. It includes tabulated results for various Reynolds numbers. Figure 3 shows the velocity component u along the vertical line through the geometric centre of cavity for Re=100. Figure 4 shows the velocity component v along the horizontal line through the geometric centre of cavity for Re=100. It is demonstrated in the figures that very good agreements have been achieved through the comparison of the present model with those obtained by Ghia at el. [1].

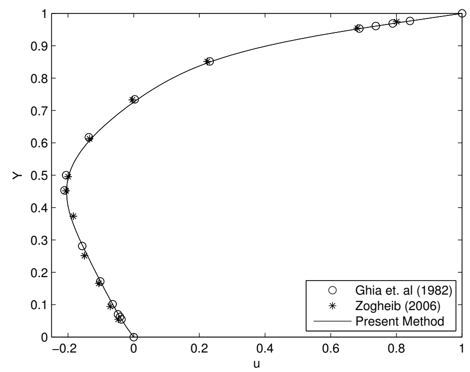

Figure 3: Velocity Along the Vertical Line Through the Geometric Centre of Cavity for Re=100

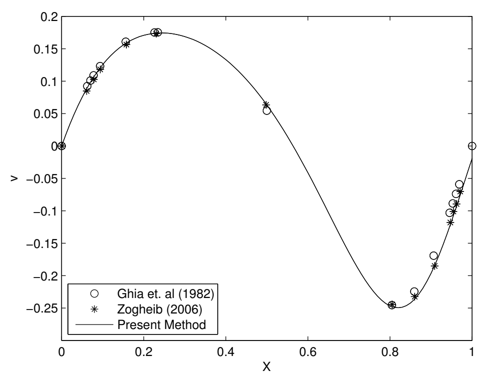

Figure 4: Velocity Along the Horizontal Line Through the Geometric Centre of Cavity for Re=100

The velocity contour of the cavity flow for Re=100 is shown in Figure 5.

5 Conclusions

An improved finite difference method for solving the two-dimensional, steady, incompressible, laminar, viscous flow equations on a staggered grid is presented for the validation of the fundamental lid-driven cavity flow problem. Good agreements have been achieved for the benchmark test.

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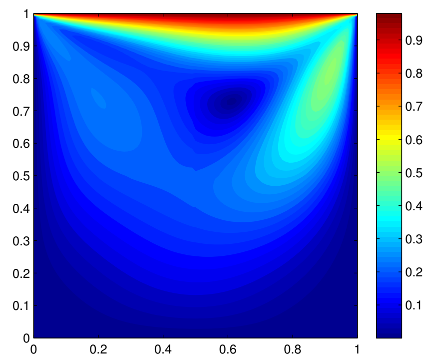

Figure 5: Velocity Contour of the Cavity Flow for Re=100

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