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AN INTEGRAL EQUATION METHOD FOR CONFORMAL MAPPING OF DOUBLY CONNECTED REGIONS INVOLVING THE KERZMAN-STEIN KERNEL

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Abstract. *We present an integral equation method for conformal mapping of doubly connected regions onto a unit disc with circular slit of radius $\mu < 1$. Our theoretical development is based on the boundary integral equation for conformal mapping of doubly connected region derived by Murid and Razali [14]. In this paper, using the boundary relationship satisfied by the mapping function, a related system of Fredholm integral equation is constructed, provided μ is assume known. For numerical experiment, the integral equation is discretized which leads to a system of linear equations. Numerical implementation on a circular annulus is also presented.*

Keyword: *Conformal mapping, Integral equation, Doubly connected region, Kerzman-Stein kernel*

1. Introduction

Numerical conformal mapping of multiply connected regions are presently still a subject of interest. Every region of connectivity p can be mapped conformally on each of the five canonical regions [2, 3, 15]. They are the disc with concentric circular slits, an annulus with concentric circular slits, the circular slit region, the radial slit region and the parallel slit region. In particular, if Ω is a multiply connected regions of connectivity (p + 1) inside the unit disc $|z| < 1$ where $\Gamma = |z| = 1$ is the boundary component of Ω, then there exists a univalent analytic function $w = f(z)$ in Ω such that (*i*) it maps Ω conformally onto a region G inside the unit disc $|w| < 1$ which has p circular slits centered at $w = 0$ and (*ii*) it maps the unit circle $|z| = 1$ conformally onto a unit circle $|w| = 1$. The images of the circular slits are traversed twice [5,10].

Several methods for conformal mapping of multiply connected regions have been proposed in the literature [4, 7, 9, 12, 14, 18, 19, 20, 22, 23]. One of the methods is the integral equation method. Some notable ones are the integral equations of Warschawski, Gerschgorin, and Symm. All these integral equations are extensions of those maps for simply connected regions. However, there are two recently derived integral equations for conformal mapping of simply connected regions which have no analogue for the doubly connected case. These are the Kerzman-Stein-Trummer integral equation and the integral equation for the Bergman kernel as derived in Kerzman and Trummer [8], Henrici [5] and Razali et al. [21]. An effort for such extension has been given by Murid and Razali [14] but without any numerical experiment. Conformal mappings of doubly connected regions onto an annulus involving the Kerzman-Stein kernel are also encountered in [13, 16, 17].

In this paper, we adapted the works in [14, 16] to construct an integral equation involving the Kerzman-Stein kernel for conformal mapping of doubly connected regions onto a unit disc with circular slit of radius $\mu <$ 1. For numerical experiment, the integral equation is discretized which leads to a system of linear equation provided μ is known. A numerical example is reported for a circular annulus as a test region.

2. The boundary integral equation for conformal mapping of doubly connected regions involving the Kerzman-Stein kernel

Let Γ_0 and Γ_1 be two smooth Jordan curves in the complex z-plane such that Γ_1 lies in the interior of Γ_0. Denote by Ω the finite doubly connected regions with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Let $w = f(z)$ be the analytic function which maps conformally Ω onto a unit disc with a circular slit of radius $\mu < 1$. The mapping function f is determined up to a factor of modulus 1. The function f could be made unique by prescribing that

$$f(a) = 0\,,\ \ f'(a) > 0\,,$$

where $a \in \Omega$ is a fixed point.

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The boundary value of f can be represented in form

$$f(z_0(t)) = e^{i\theta_0(t)}, \qquad \Gamma_0 : z = z_0(t), \quad 0 \le t \le \beta_0, \tag{1}$$

$$f(z_1(t)) = \mu e^{i\theta_1(t)}, \qquad \Gamma_1 : z = z_1(t), \quad 0 \le t \le \beta_1, \tag{2}$$

where $\theta_0(t)$ and $\theta_1(t)$ are the boundary correspondence functions of Γ_0 and Γ_1 respectively.

The unit tangent to Γ at is denoted $z(t)$ by $T(z(t)) = z'(t)/|z'(t)|$. Thus it can be shown that

$$f(z_0(t)) = \frac{1}{i} T(z_0(t)) \frac{f'(z_0(t))}{|f'(z_0(t))|}, \tag{3}$$

$$f(z_1(t)) = \frac{\mu}{i} T(z_1(t)) \frac{f'(z_1(t))}{|f'(z_1(t))|}. \tag{4}$$

The boundary relationships (3) and (4) can be combined as

$$f(z) = \frac{|f(z)|}{i} T(z) \frac{f'(z)}{|f'(z)|}, \qquad z \in \Gamma, \tag{5}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$. Based on Murid and Razali [14], it can be shown that the mapping function f of a doubly connected region satisfies the integral equation

$$\sqrt{f'(z)} + \int_\Gamma A(z,w)\sqrt{f'(w)}\,|dw|$$

$$= -i\,|f(z)|\,\overline{T(z)}\left[\operatorname*{Res}_{w=a} \frac{\sqrt{f'(w)}}{(w-z)f(w)}\right]^- - i\,\overline{T(z)}(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{\sqrt{f'(w)}}{(w-z)f(w)}dw\right]^-, \quad z \in \Gamma, \tag{6}$$

where

$$A(z,w) = \begin{cases} \overline{H(w,z)} - H(z,w), & \text{if } w,z \in \Gamma, w \ne z, \\ 0, & \text{if } w = z \in \Gamma, \end{cases} \tag{7}$$

and

$$H(w,z) = \frac{1}{2\pi i}\frac{T(z)}{z-w}, \qquad w \in \Omega \cup \Gamma, z \in \Gamma, w \ne z. \tag{8}$$

The minus sign in the superscript denotes complex conjugate and where

$$\Gamma_2 = \begin{cases} -\Gamma_1, & \text{if } z \in \Gamma_0, \\ \Gamma_0, & \text{if } z \in \Gamma_1. \end{cases}$$

The kernel A is known as the Kerzman-Stein kernel [8], and is smooth and skew-Hermitian. The kernel H is usually referred to as the Cauchy kernel. The uniqueness of the solution for the integral equation is guaranteed from the fact that the kernel $A(z, w)$ is skew-hermitian on $\Gamma \times \Gamma$ and therefore has a purely imaginary spectrum. However, no numerical experiments are reported in Murid and Razali [14] because the integral equation is not in the form of Fredholm integral equation and evaluation of the right-hand side is yet undetermined.

Since $f(a) = 0$ and $f'(a) > 0$, by means of residue theorem [6, 11], we have

$$\operatorname*{Res}_{w=a} \frac{\sqrt{f'(w)}}{(w-z)f(w)} = \frac{\sqrt{f'(a)}}{(a-z)f'(a)} = \frac{1}{(a-z)\sqrt{f'(a)}} \tag{9}$$

Thus integral equation (6) becomes

$$\sqrt{f'(z)} + \int_{\Gamma} A(z,w)\sqrt{f'(w)}\,|\,dw\,|$$

$$= -i\,|\,f(z)\,|\,\overline{T(z)}\left[\frac{1}{(a-z)\sqrt{f'(a)}}\right]^{-} - i\,\overline{T(z)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sqrt{f'(w)}}{(w-z)\,f(w)}dw\right]^{-},\ z\in\Gamma. \quad (10)$$

Multiply (10) by $\sqrt{f'(a)}$, we get

$$\sqrt{f'(z)f'(a)} + \int_{\Gamma} A(z,w)\sqrt{f'(w)f'(a)}\,|\,dw\,|$$

$$= -i\,|\,f(z)\,|\,\frac{\overline{T(z)}}{\overline{a}-\overline{z}} - i\,\overline{T(z)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sqrt{f'(w)f'(a)}}{(w-z)\,f(w)}dw\right]^{-},\quad z\in\Gamma,$$

which implies

$$\sqrt{f'(z)f'(a)} + \int_{\Gamma} A(z,w)\sqrt{f'(w)f'(a)}\,|\,dw\,|$$

$$= 2\pi\,|\,f(z)\,|\,\overline{H(a,z)} - i\,\overline{T(z)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sqrt{f'(w)f'(a)}}{(w-z)\,f(w)}dw\right]^{-},\quad z\in\Gamma, \quad (11)$$

where

$$\overline{H(a,z)} = \frac{1}{2\pi i}\frac{\overline{T(z)}}{\overline{a}-\overline{z}}.$$

The single integral equation in (11) can be separated into a system of two integral equations given by

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma} A(z_0,w)\sqrt{f'(w)f'(a)}\,|\,dw\,|$$

$$= 2\pi\,|\,f(z_0)\,|\,\overline{H(a,z_0)} - i\,\overline{T(z_0)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{-\Gamma_1}\frac{\sqrt{f'(w)f'(a)}}{(w-z_0)\,f(w)}dw\right]^{-},\quad z_0\in\Gamma_0, \quad (12)$$

$$\sqrt{f'(z_1)f'(a)} + \int_{\Gamma} A(z_1,w)\sqrt{f'(w)f'(a)}\,|\,dw\,|$$

$$= 2\pi\,|\,f(z_1)\,|\,\overline{H(a,z_1)} - i\,\overline{T(z_1)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\sqrt{f'(w)f'(a)}}{(w-z_1)\,f(w)}dw\right]^{-},\quad z_1\in\Gamma_1. \quad (13)$$

Taking boundary relationship (5) into account, (12) and (13) become

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma} A(z_0,w)\sqrt{f'(w)f'(a)}\,|\,dw\,|$$

$$= 2\pi\overline{H(a,z_0)} - i\,\overline{T(z_0)}\,(1-\mu)\left[\frac{1}{2\pi i}\int_{-\Gamma_1}\frac{\sqrt{f'(w)f'(a)}}{(w-z_0)\left[\frac{\mu}{i}T(w)\frac{f'(w)}{|\,f'(w)\,|}\right]}dw\right]^{-},\quad z_0\in\Gamma_0, \quad (14)$$

$$\sqrt{f'(z_1)f'(a)} + \int_\Gamma A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)} - i\,\overline{T(z_1)}(1-\mu)\left[\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\sqrt{f'(w)f'(a)}}{(w-z_1)\left[\frac{1}{i}T(w)\frac{f'(w)}{|f'(w)|}\right]}dw\right]^{-}, \quad z_1 \in \Gamma_1. \tag{15}$$

Using $|f'(w)| = \sqrt{f'(w)}\overline{\sqrt{f'(w)}}$ and $T(w)|dw| = dw$, the two integral equations (14) and (15) become

$$\sqrt{f'(z_0)f'(a)} + \int_\Gamma A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\overline{H(a,z_0)} + \frac{1}{2\pi i\mu}\overline{T(z_0)}(1-\mu)\int_{-\Gamma_1}\frac{\sqrt{f'(w)f'(a)}}{\overline{w}-\overline{z}_0}|dw|, \quad z_0 \in \Gamma_0, \tag{16}$$

$$\sqrt{f'(z_1)f'(a)} + \int_\Gamma A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)} + \frac{1}{2\pi i}\overline{T(z_1)}(1-\mu)\int_{\Gamma_0}\frac{\sqrt{f'(w)f'(a)}}{\overline{w}-\overline{z}_1}|dw|, \quad z_1 \in \Gamma_1. \tag{17}$$

Since $\Gamma = \Gamma_0 \cup \Gamma_1$, the equation (16) and (17) can be written as

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma_0} A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\overline{H(a,z_0)} + \frac{1}{2\pi i\mu}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{w}-\overline{z}_0}\sqrt{f'(w)f'(a)}\,|dw| - \frac{1}{2\pi i}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{w}-\overline{z}_0}\sqrt{f'(w)f'(a)}\,|dw|, \quad z_0 \in \Gamma_0, \tag{18}$$

$$\sqrt{f'(z_1)f'(a)} + \int_{\Gamma_0} A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)} + \frac{1}{2\pi i}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{\overline{w}-\overline{z}_1}\sqrt{f'(w)f'(a)}\,|dw| - \frac{\mu}{2\pi i}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{\overline{w}-\overline{z}_1}\sqrt{f'(w)f'(a)}\,|dw|, \quad z_1 \in \Gamma_1. \tag{19}$$

Applying definitions (7) and (8) to $A(z_0,w)$ in $\int_{-\Gamma_1}$ of equation (18) and to $A(z_1,w)$ in $\int_{\Gamma_0}$ of equation (19), we obtain

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma_0} A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1}\frac{1}{2\pi i}\left[\frac{\overline{T(z_0)}}{(\overline{w}-\overline{z}_0)} - \frac{T(w)}{w-z_0}\right]\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\overline{H(a,z_0)} + \frac{1}{2\pi i\mu}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{w}-\overline{z}_0}\sqrt{f'(w)f'(a)}\,|dw| - \frac{1}{2\pi i}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{w}-\overline{z}_0}\sqrt{f'(w)f'(a)}\,|dw|, \quad z_0 \in \Gamma_0, \tag{18}$$

$$\sqrt{f'(z_1)f'(a)} + \int_{\Gamma_0}\frac{1}{2\pi i}\left[\frac{\overline{T(z_1)}}{(\overline{w}-\overline{z}_1)} - \frac{T(w)}{w-z_1}\right]\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)} + \frac{1}{2\pi i}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{(\overline{w}-\overline{z}_1)}\sqrt{f'(w)f'(a)}\,|dw| - \frac{\mu}{2\pi i}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{\overline{w}-\overline{z}_1}\sqrt{f'(w)f'(a)}\,|dw|, \quad z_1 \in \Gamma_1. \tag{19}$$

After some cancellations, we get

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma_0} A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw| + \frac{1}{2\pi i}\int_{-\Gamma_1} \frac{T(w)}{w-z_0}\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\overline{H(a,z_0)} + \frac{1}{2\pi i\mu}\int_{-\Gamma_1} \frac{\overline{T(z_0)}}{(\overline{w}-\overline{z}_0)}\sqrt{f'(w)f'(a)}\,|dw|, \quad z_0 \in \Gamma_0, \tag{20}$$

$$\sqrt{f'(z_1)f'(a)} - \frac{1}{2\pi i}\int_{\Gamma_0} \frac{T(w)}{w-z_1}\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)} - \frac{\mu}{2\pi i}\int_{\Gamma_0} \frac{\overline{T(z_1)}}{(\overline{w}-\overline{z}_1)}\sqrt{f'(w)f'(a)}\,|dw|, \; z = z_1 \in \Gamma_1. \tag{21}$$

Rearranging (20) and (21) yields

$$\sqrt{f'(z_0)f'(a)} + \int_{\Gamma_0} A(z_0,w)\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} \frac{1}{2\pi i}\left[\frac{\overline{T(z_0)}}{\mu(\overline{w}-\overline{z}_0)} - \frac{T(w)}{w-z_0}\right]\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\overline{H(a,z_0)}, \qquad z_0 \in \Gamma_0, \tag{22}$$

$$\sqrt{f'(z_1)f'(a)} + \int_{\Gamma_0} \frac{1}{2\pi i}\left[\frac{\mu\overline{T(z_1)}}{\overline{w}-\overline{z}_1} - \frac{T(w)}{w-z_1}\right]\sqrt{f'(w)f'(a)}\,|dw| - \int_{-\Gamma_1} A(z_1,w)\sqrt{f'(w)f'(a)}\,|dw|$$

$$= 2\pi\mu\overline{H(a,z_1)}, \qquad z_1 \in \Gamma_1. \tag{23}$$

Defining

$$\sqrt{f'(z)f'(a)} = \begin{cases} U(z_0(t),a), & \text{if } z_0(t) \in \Gamma_0, \\ U(z_1(t),a), & \text{if } z_1(t) \in \Gamma_1, \end{cases}$$

$$B(z,w) = \frac{1}{2\pi i}\left[\frac{\overline{T(z)}}{\mu(\overline{w}-\overline{z})} - \frac{T(w)}{w-z}\right],$$

$$D(z,w) = \frac{1}{2\pi i}\left[\frac{\mu\overline{T(z)}}{\overline{w}-\overline{z}} - \frac{T(w)}{w-z}\right],$$

(22) and (23) can be written as

$$U(z_0,a) + \int_{\Gamma_0} A(z_0,w)U(w,a)\,|dw| - \int_{-\Gamma_1} B(z_0,w)U(w,a)\,|dw| = 2\pi\,\overline{H(a,z_0)}, \qquad z_0 \in \Gamma_0, \tag{24}$$

$$U(z_1,a) + \int_{\Gamma_0} D(z_1,w)U(w,a)\,|dw| - \int_{-\Gamma_1} A(z_1,w)U(w,a)\,|dw| = 2\pi\mu\,\overline{H(a,z_1)}, \qquad z_1 \in \Gamma_1. \tag{25}$$

If μ is assume known, then (24) and (25) form a system of Fredholm integral equations of the second kind that has a unique solution.

3. Numerical implementation

Using parametric representations $z_0(t)$ of Γ_0 for $t : \le t \le \beta_0$ and $z_1(t)$ of Γ_1 for $t : 0 \le t \le \beta_1$, the equations (24) and (25) become

$$U(z_0(t),a) + \int_0^{\beta_0} A(z_0(t),z_0(s))U(z_0(s),a)\,|z'_0(s)|\,ds$$

$$-\int_0^{\beta_1} B(z_0(t), z_1(s))\, U(z_1(s), a)\, | z'_1(s) |\, ds = 2\pi\, \overline{H(a, z_0)}, \quad z_0 \in \Gamma_0, \tag{26}$$

$$U(z_1(t), a) + \int_0^{\beta_0} D(z_1(t), z_0(s))\, U(z_0(s), a)\, | z'_0(s) |\, ds$$
$$-\int_0^{\beta_1} A(z_1(t), z_1(s))\, U(z_1(s), a)\, | z'_1(s) |\, ds = 2\pi\mu\, \overline{H(a, z_1)}, \quad z_1 \in \Gamma_1. \tag{27}$$

Multiply (26) and (27) respectively by $|z_0'(t)|^{1/2}$ and $|z_1'(t)|^{1/2}$ gives

$$|z_0'(t)|^{1/2} U(z_0(t), a) + \int_0^{\beta_0} |z_0'(t)|^{1/2} |z_0'(s)|^{1/2} A(z_0(t), z_0(s))\, U(z_0(s), a) |z_0'(s)|^{1/2} ds$$
$$-\int_0^{\beta_1} |z_0'(t)|^{1/2} |z_1'(s)|^{1/2} B(z_0(t), z_1(s))\, U(z_1(s), a) |z_1'(s)|^{1/2} ds = 2\pi |z_0'(t)|^{1/2}\, \overline{H(a, z_0(t))}, \; z_0 \in \Gamma_0, \tag{28}$$

$$|z_1'(t)|^{1/2} U(z_1(t), a) + \int_0^{\beta_0} |z_1'(t)|^{1/2} |z_0'(s)|^{1/2} D(z_1(t), z_0(s))\, U(z_0(s), a) |z_0'(s)|^{1/2} ds$$
$$-\int_0^{\beta_1} |z_1'(t)|^{1/2} |z_1'(s)|^{1/2} A(z_1(t), z_1(s))\, U(z_1(s), a) |z_1'(s)|^{1/2} ds = 2\pi\mu |z_1'(t)|^{1/2}\, \overline{H(a, z_1(t))}, \; z_1 \in \Gamma_1. \tag{29}$$

Defining

$$\phi_0(t) = |z_0'(t)|^{1/2}\, U(z_0(t), a),$$

$$\phi_1(t) = |z_1'(t)|^{1/2}\, U(z_1(t), a),$$

$$\gamma_0(t) = 2\pi |z_0'(t)|^{1/2}\, \overline{H(a, z_0(t))},$$

$$\gamma_1(t) = 2\pi\mu |z_1'(t)|^{1/2}\, \overline{H(a, z_1(t))},$$

$$K_{00}(t_0, s_0) = |z_0'(t)|^{1/2} |z_0'(s)|^{1/2} A(z_0(t), z_0(s)),$$

$$K_{01}(t_0, s_1) = |z_0'(t)|^{1/2} |z_1'(s)|^{1/2} B(z_0(t), z_1(s)),$$

$$K_{10}(t_1, s_0) = |z_1'(t)|^{1/2} |z_0'(s)|^{1/2} D(z_1(t), z_0(s)),$$

$$K_{11}(t_1, s_1) = |z_1'(t)|^{1/2} |z_1'(s)|^{1/2} A(z_1(t), z_1(s)).$$

Thus equations (28) and (29) can be briefly written as

$$\phi_0(t) + \int_0^{\beta_0} K_{00}(t_0, s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{01}(t_0, s_1)\phi_1(s)ds = \gamma_0(t), \quad z_0(t) \in \Gamma_0, \tag{30}$$

$$\phi_1(t) + \int_0^{\beta_0} K_{10}(t_1, s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{11}(t_1, s_1)\phi_1(s)ds = \gamma_1(t), \quad z_1(t) \in \Gamma_1. \tag{31}$$

Note that the kernel $K_{00}(t_0, s_0)$ and $K_{11}(t_1, s_1)$ preserve the skew-Hermitian properties.

Since the functions ϕ, γ, and K are β-periodic, an appealing procedure for solving (30) and (31) numerically is using the Nyström's method with the trapezoidal rule [1]. The trapezoidal rule is the most accurate method for integrating periodic functions numerically. We choose $\beta_0 = \beta_1 = 2\pi$ and n equidistant collocation points $t_i = (i-1)\beta_0 / n$, $1 \le i \le n$ on Γ_0 and m equidistant collocation points $t_{\tilde{i}} = (\tilde{i} - 1)\beta_1 / m$, $1 \le \tilde{i} \le m$ on Γ_1. Applying the Nyström's method with trapezoidal rule to discretize (30) and (32), gives

$$\phi_0(t_i) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{00}(t_i, t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{\tilde{j}=1}^{m} K_{01}(t_i, t_{\tilde{j}})\phi_1(t_{\tilde{j}}) = \gamma_0(t_i), \tag{32}$$

$$\phi_1(t_{\tilde{i}}) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{10}(t_{\tilde{i}}, t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{\tilde{j}=1}^{m} K_{11}(t_{\tilde{i}}, t_{\tilde{j}})\phi_1(t_{\tilde{j}}) = \gamma_1(t_{\tilde{i}}), \tag{33}$$

Equations (32) and (33) lead to a system of $(n + m)$ linear complex equations in n unknowns $\phi_0(t_i)$ and m unknowns $\phi_1(t_{\tilde{i}})$.

By defining the matrices

$$B_{ij} = \frac{\beta_0}{n} K_{00}(t_i, t_j)\,,\; C_{i\tilde{j}} = \frac{\beta_1}{m} K_{01}(t_i, t_{\tilde{j}})\,,\; E_{\tilde{i}j} = \frac{\beta_0}{n} K_{10}(t_{\tilde{i}}, t_j)\,,\; D_{\tilde{i}\tilde{j}} = \frac{\beta_1}{m} K_{11}(t_{\tilde{i}}, t_{\tilde{j}})\,,$$

$$x_{0i} = \phi_0(t_i)\,,\; x_{1\tilde{i}} = \phi_1(t_{\tilde{i}})\,,\; \gamma_{0i} = \gamma_0(t_i)\,,\; \gamma_{1\tilde{i}} = \gamma_1(t_{\tilde{i}})\,,$$

the system of equations (32) and (33) can be written as $n + m$ by $n + m$ system of linear equations

$$[I_{nn} + B_{nn}]x_{0n} - C_{nm}x_{1m} = \gamma_{0n}\,, \tag{34}$$

$$E_{mn}x_{0n} + [I_{mm} - D_{mm}]x_{1m} = \gamma_{1m}\,. \tag{35}$$

The result in matrix form for the system of equations (34) and (35) is

$$\begin{pmatrix} I_{nn} + B_{nn} & \vdots & -C_{nm} \\ \cdots & \cdots & \cdots \\ E_{mn} & \vdots & I_{mm} - D_{mm} \end{pmatrix} \begin{pmatrix} x_{0n} \\ \cdots \\ x_{1m} \end{pmatrix} = \begin{pmatrix} \gamma_{0n} \\ \cdots \\ \gamma_{1m} \end{pmatrix}. \tag{36}$$

Defining

$$A = \begin{pmatrix} I_{nn} + B_{nn} & \vdots & -C_{nm} \\ \cdots & \cdots & \cdots \\ E_{mn} & \vdots & I_{mm} - D_{mm} \end{pmatrix}\,,\quad \boldsymbol{x} = \begin{pmatrix} x_{0n} \\ \cdots \\ x_{1m} \end{pmatrix},\quad \boldsymbol{y} = \begin{pmatrix} \gamma_{0n} \\ \cdots \\ \gamma_{1m} \end{pmatrix},$$

the $(n + m) \times (n + m)$ system can be written briefly as $A\boldsymbol{x} = \boldsymbol{y}$. Separating A, $\boldsymbol{x}$ and $\boldsymbol{y}$ in terms of the real and imaginary parts, the system can be written as

$$\mathrm{Re}\, A\, \mathrm{Re}\, \boldsymbol{x} - \mathrm{Im}\, A\, \mathrm{Im}\, \boldsymbol{x} + i\,(\mathrm{Im}\, A\, \mathrm{Re}\, \boldsymbol{x} + \mathrm{Re}\, A\, \mathrm{Im}\, \boldsymbol{x}) = \mathrm{Re}\, \boldsymbol{y} + i\, \mathrm{Im}\, \boldsymbol{y}. \tag{37}$$

The single $(n + m) \times (n + m)$ complex linear system (37) can also written as $2(n + m) \times 2(n + m)$ system matrix involving the real (Re) and imaginary (Im) of the unknown functions, i.e.,

$$\begin{pmatrix} \mathrm{Re}\, A & \vdots & -\mathrm{Im}\, A \\ \cdots & \cdots & \cdots \\ \mathrm{Im}\, A & \vdots & \mathrm{Re}\, A \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, x \\ \cdots \\ \mathrm{Im}\, x \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, y \\ \cdots \\ \mathrm{Im}\, y \end{pmatrix}. \tag{38}$$

Therefore, the linear system (38) can be solved simultaneously for the unknown functions,

$$\phi_0(t) = |z_0'(t)|^{1/2} \sqrt{f'(z_0(t))f'(a)}\,, \tag{39}$$

$$\phi_1(t) = |z_1'(t)|^{1/2} \sqrt{f'(z_1(t))f'(a)}\,. \tag{40}$$

The boundary correspondence functions $\phi_0(t)$ and $\phi_1(t)$ are then computed approximately by the formulas

$$\theta_0(t) = \mathrm{Arg}\, f(z_0(t)) \approx \mathrm{Arg}\,(-iz_0'(t)\phi_0^2(t))\,, \tag{41}$$

$$\theta_1(t) = \mathrm{Arg}\, f(z_1(t)) \approx \mathrm{Arg}\,(-iz_1'(t)\phi_1^2(t))\,. \tag{42}$$

4. Numerical results

For numerical experiment, we have used frame of circular annulus $A = \{z : r < |z| < 1\}$, $r = q = e^{-\pi\tau}$, $\tau > 0$ as a test region. The exact mapping function is given by [24, p. 362]

$$f(z) = -e^{2\sigma} \frac{\theta_4\left(\frac{1}{2i}\log z + \frac{i\pi\tau}{2} - i\sigma\right)}{\theta_4\left(\frac{1}{2i}\log z + \frac{i\pi\tau}{2} + i\sigma\right)} \tag{43}$$

with $\mu = e^{-2\sigma}$ and θ_4 being the Jacobi Theta-functions. We have chosen τ = 0.50 and σ = 0.20. Since $\theta_4(\pi\tau i/2) = 0$ [25], this implies $a = e^{-2\sigma} = \mu$. Figure 1 shows the region and image based on our method. The result for sub-norm error between the exact boundary correspondence functions $\theta_0(t)$, $\theta_1(t)$ and the computed boundary correspondence functions $\theta_{0n}(t)$, $\theta_{0m}(t)$ is shown in Table 1. All the computations are done using MATHEMATICA package [26] in single precision (16 digit machine precision).

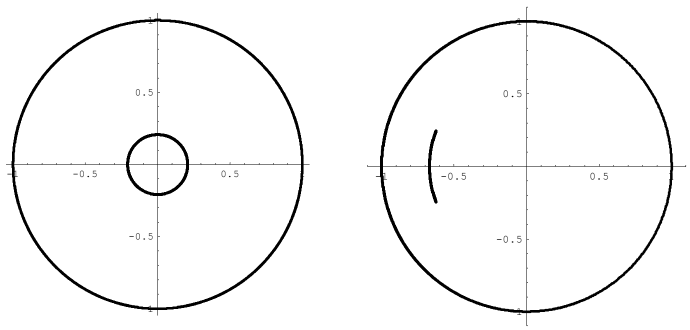

Figure 1. Conformal mapping of a circular annulus to the unit disc with a circular slit : τ = 0.50, σ = 0.20, $r = e^{-\pi\tau}$, $a = \mu = e^{-2\sigma}$

Table 1. Error Norm

$n = m$	$\|\theta_0(t) - \theta_{0n}(t)\|_\infty$	$\|\theta_1(t) - \theta_{1m}(t)\|_\infty$
16	1.1 (-03)	1.1 (-0.2)
32	1.9 (-06)	3.7 (-0.5)
64	5.2 (-12)	2.0 (-10)
128	4.0 (-15)	3.6 (-15)

5. Conclusion

In this paper we have constructed a system of integral equations for numerical conformal mapping from a doubly connected regions onto a unit disc with a concentric circular slit. The system involved the Kerzman-Stein kernel and is Fredholm type provided μ is known. The advantage of this method is that it calculates the boundary correspondence functions simultaneously. The numerical example illustrates that the present method can be used to produce approximations of high accuracy provided μ is known.

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