Menemui Matematik (Discovering Mathematics)
Vol. 32, No. 2: 35– 41 (2010)

On Some Problems in Group Theory of Probabilistic Nature

[1]Nor Muhainiah Mohd Ali and [2]Nor Haniza Sarmin
Department of Mathematics, Faculty of Science,
Universiti Teknologi Malaysia,
81310 UTM Johor Bahru, Johor.
Email: [1]muhainiah9119@yahoo.com, [2]nhs@mel.fs.utm.my

ABSTRACT

The determination of the abelianness of a nonabelian group has been introduced for symmetric groups by Erdos and Turan in 1968. In 1973, Gustafson did the same for finite groups while MacHale determined the abelianness for finite rings in 1974. Basic probability theory will be used in connection with group theory. This paper will focus on the 2-generator 2-groups of nilpotency class 2 based on the classification that has been done by Kappe *et.al* in 1999. In this paper some results on $P_n(G)$, the probability that the n^{th} power of a random element in a group G commutes with another random element from the same group, will be presented.

INTRODUCTION

In the past 20 years, and particularly during the last decade, there has been a growing interest in the use of probability in group theory. There has been an intensive study to find the degree of abelianness of a group. This theory has been introduced for symmetric groups, finite groups and finite rings. In 1968, Erdos and Turan published a paper on the 'statistics' of the symmetric group S_n. The quotient,

$$P(G) = \frac{\text{Number of ordered pairs } (x,y) \in GxG \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x,y) \in GxG}$$

$$= \frac{|\{(x,y) \in GxG\, xy = |yx\}\|}{|G|^2}$$

has been used by several authors to find the probability that a random pair of elements in the group G commute. Clearly, this quotient is equal to 1 if and only if the group is abelian. Such probability can be associated with every finite group, and this number can be considered as a measure how abelian a nonabelian group is. The same concept was then used by Gustafson (1973) and MacHale (1974) to find the probability for finite groups in general and they showed that this probability cannot be arbitrarily close to 1 if G is a finite nonabelian group. In fact, it will always be less than or equal to 5/8. In this paper, we will only consider the nonabelian groups. The classification of 2-generator 2-groups of nilpotency class 2 has been introduced by Kappe *et. al*. in 1999. The classification has then been modified by Magidin (2006). Since the latter classification is easier to use, it will be used throughout this research. The probability that the n^{th} power of a random element commutes with another random element from the same group, $P_n(G)$, is defined as

.

$$P_n(G) = \frac{|\{(x,y)\in GxG xy = |yx\}\|}{|G^{2}_{\dagger}}$$

In this paper, $P_n(G)$ for some 2-generator 2-groups of nilpotency class 2 will be presented.

SOME PREPARATORY RESULTS

The classification of 2-generator 2-groups of nilpotency class 2 done by Kappe *et. al.* in 1999 is given in Theorem 2.1. The modified classification due to Magidin (2006) is given in Theorem 2.2.

Theorem 2.1 [Kappe *et al.*, 1999]

Let G be a finite nonabelian 2-generator 2-groups of class 2. Then G is isomorphic to exactly one group of the following four types:

(2.1.1) $G \cong (\langle c \rangle \text{ x} \langle a \rangle) \rtimes \langle b \rangle$ where [a,b] = c, [a,c] = [b, c] = 1. $\| a \| = 2^{\alpha}, \| b \| = 2^{\beta}$, $|c| = 2\gamma, \alpha, \beta, \gamma \epsilon \square$, and $\alpha \le \beta \ge \gamma \ge 1$.

(2.1.2) $G \cong \langle a \rangle \rtimes \langle b \rangle$ where [a, b] = $\alpha^{2^{\alpha-\gamma}}$, $|a| = 2^{\alpha}, |b| = 2^{\beta}, |[a,b]| = 2^{\gamma}, \alpha, \alpha, \gamma \epsilon \square$ and $\alpha \ge \beta, \alpha + \beta > 3, \alpha \ge 2\gamma, \beta \ge \gamma \ge 1$.

(2.1.3) $G \cong (\langle c \rangle \text{x} \langle a \rangle) \rtimes \langle b \rangle$ where [a, b] = $\alpha^{2^{\alpha-\gamma}}$c, [c, b] = $\alpha^{-2^{\alpha-\gamma}} c^{-2^{\alpha-\gamma}}, |a| = 2^{\alpha}$ $|b| = 2\beta, |c| = 2^{\sigma}, |[a,b]| = 2^{\gamma}, \alpha, \beta, \gamma\sigma \epsilon \square$ and $\gamma > \sigma \ge 1,\ \alpha + \sigma \ge 2\gamma, \alpha \ge \beta, \beta \ge \gamma$.

(2.1.4) $G \cong (\langle c \rangle \text{x} \langle a \rangle) \langle b \rangle$, where $|a| = |b| = 2^{\gamma+1}, |[a,b]| = 2^{\gamma}, |c| = 2^{\gamma-1}[\alpha, b] = a^2 c, [c,b]$ $= a^{-4} c^{-2} -, a^{2\gamma} = b^{2\gamma}, \gamma \epsilon \square$

Note that the extra condition $\alpha + \beta > 3$ is added in (2.1.2) to avoid that the Dihedral group of order 8 is not included in both (2.1.1) and (2.1.2). Instead, it is characterized to be of type (2.1.1). The condition $\alpha + \beta > 5$ is added so that the groups of order 64 do not appear in both 2.1.1 and 2.1.3.

Theorem 2.2 [Magidin, 2006]

Let G be a finite nonabelian 2-generator 2-groups of nilpotency class 2. Then G is isomorphic to exactly one group of the following types:

(2.2.1) $G \cong \langle a, b \rangle | a^{2\alpha} = b^{2\beta} = [a,b]^{2\gamma} = [a,b,a] = a,b,b] = e \rangle$, where $\alpha, \beta, \gamma \epsilon \square$ satisfying $\alpha \ge \beta \ge \gamma$

(2.2.2) $G \cong \langle a, b \backslash \alpha^{2^{\alpha}} = b^{2^{\beta}} = [a,b,a] = a,b,b] = e, a^{2^{\alpha+\sigma-\gamma}} [a,b]^{2^{\sigma}} \rangle$, with $\alpha, \beta, \gamma, \sigma \epsilon \square$ satisfying $\beta \ge \gamma > \sigma > 0, \alpha + \sigma \ge 2\gamma$ and $\alpha + \beta > 3$.

(2.2.3) $G \cong \langle a, b \backslash \alpha^{2^{\gamma+1}} = [a, b^{2^{\gamma}} = [a,b,a] = [a,b,b] = e, a^{2^{\gamma}} = b^{2^{\gamma}} = [a,b]^{2^{\gamma+1}} \rangle$, with $\gamma \epsilon \square$.

Note that [a, b, c] = [[a, b], c].

In the case $n = 1$, $P(G)$ can be computed using the following result.

Theorem 2.3 [MacHale, 1974]

Let the finite group G has $k(G)$ conjugacy classes. Then $P(G)=\dfrac{k(G)}{|G|}$.

THE USE OF GROUPS, ALGORITHMS AND PROGRAMMING (GAP) SOFTWARE

Recall from Section 1 that $P_n(G)$ denotes the probability that the n^{th} power of a random element in G commutes with another random element from the same group. In this paper, we will present some values of $P_n(G)$, where G is a 2-generator 2-group of nilpotency class 2 of less than or equal to 64. We will also include the results in the case where n=1. The Groups, Algorithms & Programming (GAP) Software is used to identify the groups that fulfill the classification and to confirm the number of conjugacy classes of a group. GAP is also used to find the Cayley Table for the groups order and 2^6 and to find all the elements in the groups of order 2^3, 2^4, 2^5 and 2^6. Using GAP, the groups have been signed with ID of the groups. The identity of each group in GAP is denoted by the symbol [a ,b], where a is the order of the group and b is the b^{th} group defined by GAP of order a [Ahmad and Sarmin, 2004]. A group of certain order is obtained by putting suitable values of the parameters α, β, σ and γ in the classification. Theorem 2.2 by Magidin (2006) and Table 1 in Ahmad and Sarmin (2004) are used in finding the groups. Using Theorem 2.3, we get the values of $P(G)$ for all nonabelian 2-generator 2-groups of nilpotency class 2 up to order 64, given in Table 1.

Table 1: P(G) for all nonabelian 2-generator 2-groups of nilpotency class 2 of up to order 64

$\lvert G \rvert$	Type of Classification	Id Group by GAP	Conjugacy Class of G	P(G)
8	Type 2.1.1	D_4, [8,3]	5	5/8
	Type 2.1.2	No Group	-	-
	Type 2.1.3	No Group	-	-
	Type 2.1.4	Q, [8,4]	5	5/8
16	Type 2.1.1	G_{4,4}, [16,3] 16	10	5/8
	Type 2.1.2	[16,4]	10	5/8
	Type 2.1.2	Modular, [16,6]	10	5/8
	Type 2.1.3	No Group	-	-
	Type 2.1.4	No Group	-	-
32	Type 2.1.1	[32,2]	20	5/8
	Type 2.1.1	[32,5]	20	5/8
	Type 2.1.2	[32,4]	20	5/8
	Type 2.1.2	[32,12]	20	5/8
	Type 2.1.2	[32,17]	20	5/8
	Type 2.1.3	No Group	-	-
	Type 2.1.4	No Group	-	-
64	Type 2.1.1	[64,17]	22	11/32
	Type 2.1.1	[64,18]	40	5/8
	Type 2.1.1	[64,29]	40	5/8
	Type 2.1.2	[64,3]	40	5/8
	Type 2.1.2	[64,27]	40	5/8
	Type 2.1.2	[64,28]	22	11/32
	Type 2.1.2	[64,44]	40	5/8
	Type 2.1.2	[64,51]	40	5/8
	Type 2.1.3	No Group	-	-
	Type 2.1.4	[64,19]	22	11/32

SOME RESULTS ON $P_N(G)$, G IS A 2-GENERATOR 2-GROUP OF NILPOTENCY CLASS 2

In this section, some theorems on P(G), that is the probability that the n^{th} power of a random element in a group G commutes with another random element from the same group will be presented. MacHale (1974) used the 0,1-Table (or symmetrical Cayley Table) to find the probability that two elements commute in a group. We define the 0,1-Table for a group G as follows: for all x, y in G, if $xy = yx$, each of the boxes corresponding to xy and yx will be assigned the number 1. Similarly, $xy \neq yx$, if the number 0 will be placed in each of these boxes. This rule will be used throughout this paper.

To find $P_n(G)$, the power of each element is gradually raised until the power n is achieved. Since the power of each element in the group G also lies in the group G itself, we can check whether the n^{th} power commutes with the other elements in the group or not using the 0,1-Table.

Theorem 4.1

Given G, a 2-generator 2-group of nilpotency class 2 of order 8, $P_n(G)$ is given as below.

$$P_n(G) = \begin{cases} \frac{5}{8}, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

Proof:

There are only 2 nonabelian 2-generator 2-groups of nilpotency class 2 of order 8, namely D_4, Dihedral Group of order 8 and Q, Quaternion Group of Order 8.

The Cayley Tables for D_4 and Q are given below:

Table 2: Cayley Table for $G \cong D_4 = \langle a, b \backslash a^4 = b^2 = 1. ba = a^3b \rangle$

.	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^2b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	e	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	e	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	e	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	e

Table 3: Cayley Table for $G \cong Q \ = \langle a, b \backslash a^4 = 1, b^2 = a^2, . \ ba = a^3 b \rangle$

.	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^2b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	e	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	e
a^2b	a^2b	ab	b	a^3b	e	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	e	a^3	a^2

From the Cayley Tables above, we get the following 0,1-Tables (i.e Table 4 and Table 5) for D_4 and Q, respectively.

Table 4: 0,1-Table for D_4

.

.	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	1	1	1	1	1	1	1	1
a	1	1	1	1	0	0	0	0
a^2	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	0	0	0
b	1	0	1	0	1	0	1	0
ab	1	0	1	0	0	1	0	1
a^2b	1	0	1	0	1	0	1	0
a^3b	1	0	1	0	0	1	0	1

Table 5: 0,1-Table for Q

.	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	1	1	1	1	1	1	1	1
a	1	1	1	1	0	0	0	0
a^2	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	0	0	0
b	1	0	1	0	1	0	1	0
ab	1	0	1	0	0	1	0	1
a^2b	1	0	1	0	1	0	1	0
a^3b	1	0	1	0	0	1	0	1

Next, we compute the n^{th} power for of all elements in D_4 and Q (refer to Table 6 and 7). Then, we use Table 4 and Table 5 to get the $P_n(G)$, given in the last row of Table 6 and 7. Thus we get the result needed.

Table 6: The n^{th} power of elements in D_4

Power of 1	Power of 2	Power of 3	Power of 4
$(e)^1 = e$	$(e)^2 = e$	$(e)^3 = e$	$(e)^4 = e$
$(a)^1 = a$	$(a)^2 = a^2$	$(a)^3 = a^3$	$(a)^4 = e$
$(a^2)^1 = a^2$	$(a^2)^2 = e$	$(a^2)^3 = a^2$	$(a^2)^4 = e$
$(a^3)^1 = a^3$	$(a^3)^2 = a^2$	$(a^3)^3 = a$	$(a^3)^4 = e$
$(b)^1 = b$	$(b)^2 = e$	$(b)^3 = b$	$(b)^4 = e$
$(ab)^1 = ab$	$(ab)^2 = e$	$(ab)^3 = ab$	$(ab)^4 = e$
$(a^2b)^1 = a^2b$	$(a^2b)^2 = e$	$(a^2b)^3 = a^2b$	$(a^2b)^4 = e$
$(a^3b)^1 = a^3b$	$(a^3b)^2 = e$	$(a^3b)^3 = a^3b$	$(a^3b)^4 = e$
$P_1(G) = 5/8$	$P_2(G) = 1$	$P_3(G) = 5/8$	$P_4(G) = 1$

Table 7: The n^{th} power of elements in Q

Power of 1	Power of 2	Power of 3	Power of 4
$(e)^1 = e$	$(e)^2 = e$	$(e)^3 = e$	$(e)^4 = e$
$(a)^1 = a$	$(a)^2 = a^2$	$(a)^3 = a^3$	$(a)^4 = e$
$(a^2)^1 = a^2$	$(a^2)^2 = e$	$(a^2)^3 = a^2$	$(a^2)^4 = e$
$(a^3)^1 = a^3$	$(a^3)^2 = a^2$	$(a^3)^3 = a$	$(a^3)^4 = e$
$(b)^1 = b$	$(b)^2 = a^2$	$(b)^3 = a^2b$	$(b)^4 = e$
$(ab)^1 = ab$	$(ab)^2 = a^2$	$(ab)^3 = a^3b$	$(ab)^4 = e$
$(a^2b)^1 = a^2b$	$(a^2b)^2 = a^2$	$(a^2b)^3 = b$	$(a^2b)^4 = e$
$(a^3b)^1 = a^3b$	$(a^3b)^2 = a^2$	$(a^3b)^3 = ab$	$(a^3b)^4 = e$
$P_1(G) = 5/8$	$P_2(G) = 1$	$P_3(G) = 5/8$	$P_4(G) = 1$

Theorem 4.2

Given G, a 2-generator 2-group of nilpotency class 2 of order 16, $P_n(G)$ is given as below.

$$P_n(G) = \begin{cases} \frac{5}{8}, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

Proof:

There are only 3 groups of 2-generator 2-groups of nilpotency class 2 of order 16, namely G_{4,4} with Id [16,3], group with Id [16,4] and Modular group with Id [16,6]. We use the same method as in the proof of Theorem 4.1 and get the result.

Theorem 4.3

Given G, a 2-generator 2-group of nilpotency class 2 of order 32, $P_n(G)$ is given as below.

$$P_n\,(G) = \begin{cases} \frac{5}{3}\,, & \text{if } n \text{ is odd} \\ \,, & \text{if } n \text{ is even} \end{cases}$$

Proof:

There are only 5 groups of 2-generator 2-groups of nilpotency class 2 of order 32, namely group with Id [32,2], [32,4], [32,5], [32,12] and [32,17]. We use the same method as in the proof of Theorem 4.1 and get the result.

Theorem 4.4

Given G, a 2-generator 2-group of nilpotency class 2 of oder 64, $P_n(G)$ is given as below.

$$P_n\,(G) = \begin{cases} \frac{5}{8}\,, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

where $P_n(G)^*$ is the same as $P(G)$ in the case $|G| = 64$ in Table 1.

Proof:

There are only 9 groups of 2-generator 2-groups of nilpotency class 2 of order 64, namely group with Id [64,17], [64,18], [64,29], [64,3], [64,27], [64,28], [64,44], [64,51] and [64,19]. We use the same method as in the proof of Theorem 4.1 and get the result.

CONCLUSION

$P_n(G)$ is defined as the probability that the n^{th} power of a random element in a group G commutes with another random element from the same group. In this paper we found the probability in the case $n = 1$ where G is a 2-generator 2-groups of nilpotency class 2 up to order 64, using the fact that the probability is the number of conjugacy classes divided by the order of the group. We also computed and presented the probability $P_n(G)$ for n in general ($n\in$ □).

REFERENCES

Ahmad, A. Sarmin, N. (2004). "Irreducible Representation of Some Two-Generator Two-Groups of Nilpotency Class Two." Proceeding 12nd National Sceince Mathematics Symposium, International Islamic University, Malaysia.

Erdos, P. and Turan, P. (1968). "On some problems of statistical group theory." *Acta Math.Acad. of Sci. Hung*, **19**: 413-435.

Gustafon, W (1973). "What is the probability that two group elements commute?" *American Mathematical Monthly*, **80**: 1031-1034.

Kappe, L-C, N. Sarmin and Visscher, M.P. (1999). "2-generator 2-groups of Class 2 and their Nonabelian Tensor Squares." *Glasgow Math. J.* **41**: 417-430.

MacHale, D. (1974), "How commutative can a non-commutative group be?" *The Mathematical Gazette*, **58**: 199-202.

Magidin, A. (2006) "Capable Two-generator Two-groups of Class Two." *arXiv: math. GR0404499*. Submitted. 169-179.