BIEBERBACH GROUPS WITH FINITE POINT GROUPS

NOR'ASHIQIN BINTI MOHD IDRUS

UNIVERSITI TEKNOLOGI MALAYSIA

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NOR'ASHIQIN BINTI MOHD IDRUS

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To

my beloved husband,

children and mother

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ABSTRACT

A Bieberbach group is a torsion free crystallographic group. It is an extension of a lattice group, which is a maximal normal free abelian group of finite rank, by a finite point group. The main objective of this research is to compute the nonabelian tensor square of Bieberbach groups with a finite nonabelian point group, in particular the dihedral group of order eight. Bieberbach groups in the Crystallographic AlgoRithms And Tables (CARAT) homepage were first explored and examples of the nonabelian tensor square of the groups were then computed by using the Groups, Algrorithms, Programming (GAP) software system. The exploration of the groups and the examples computed led to the exact characterization of the Bieberbach groups with trivial center. The centerless Bieberbach groups are interesting since they do not arise in the general construction of a Bieberbach group for a given point group. This construction has been shown to depend on the presentation of the point group. In addition, the experimental data of the computation of the nonabelian tensor square gives no insight into the structure of the tensor square such as its generators and relations. With the method developed for polycyclic groups, the nonabelian tensor square of one of the centerless Bieberbach groups with dihedral point group of order eight were manually computed. It has been demonstrated that the use of GAP helps to simplify the manual calculation. Furthermore, the computation of some homological functors of all 73 centerless Bieberbach groups with dihedral point group of order eight and of dimension at most six were explored. Lastly, some homological functors for Bieberbach groups with some other nonabelian point groups were also computed with the help of GAP.

ABSTRAK

Kumpulan Bieberbach adalah kumpulan kristolografi yang bebas kilasan. Ia adalah perluasan kepada kumpulan kekisi iaitu kumpulan abelan bebas yang normal dan maksimal, melalui kumpulan titik terhingga. Objektif utama kajian ini adalah untuk mengira kuasa dua tensor tak abelan bagi kumpulan Bieberbach dengan kumpulan titik tak abelan berperingkat terhingga, perincian kepada kumpulan titik dwihedron berperingkat lapan. Kajian ini dimulakan dengan meneroka kumpulan tersebut dalam laman *Crystallographic AlgoRithms And Tables* (CARAT) dan seterusnya pengiraan contoh-contoh kuasa dua tensor tak abelan kumpulan tersebut dibuat dengan menggunakan sistem perisian *Groups, Algrorithms and Programming* (GAP). Hasil penerokaan dan contoh-contoh pengiraan menghala ke arah pencirian kumpulan Bieberbach dengan pusat remeh. Kumpulan Bieberbach tidak berpusat adalah menarik kerana ia tidak wujud dalam pembinaan umum kumpulan Bieberbach untuk kumpulan titik yang diberi. Pembinaan ini ditunjukkan bergantung kepada persembahan kumpulan titik tersebut. Tambahan lagi, data eksperimental bagi pengiraan kuasa dua tensor tak abelan tidak memberi maklumat yang mendalam tentang struktur kumpulan tersebut seperti penjana dan perhubungannya. Dengan menggunakan kaedah kumpulan polikitaran, kuasa dua tensor tak abelan bagi salah satu kumpulan Bieberbach yang tidak berpusat dengan kumpulan titik dwihedron berperingkat lapan telah dilakukan secara manual. Penggunaan GAP ditunjukkan dalam membantu memudahkan pengiraan secara manual tersebut. Selanjutnya, pengiraan beberapa fungtor berhomologi bagi kesemua 73 kumpulan Bieberbach dengan kumpulan titik dwihedron berperingkat lapan dan berdimensi enam atau kurang dikaji. Akhir sekali, beberapa fungtor berhomologi untuk kumpulan Bieberbach dengan kumpulan titik tak abelan yang lain juga dikira dengan bantuan GAP.

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LIST OF SYMBOLS

1	-	Identity element
$Ker(f)$	-	Kernel of mapping f
$A \leq B$	-	A is a subgroup of B
$\langle a \rangle$	-	subgroup generated by an element a
$A \lhd B$	-	A is a normal subgroup of B
G/H	-	the quotient group G by H where $H \lhd G$
$A \times B$	-	direct product of A and B
x^y	-	x conjugate by y, $y^{-1}xy$
$[a, b]$	-	commutator of a and b, $a^{-1}b^{-1}ab$
$H \cong G$	-	H is isomorphic to G
$G \wedge G$	-	nonabelian exterior square of G
$H_2(G)$	-	Schur Multiplier of G
C_n	-	Cyclic group of order n
$\langle X \mid R \rangle$	-	Group presented by generators X and relators R
$H \otimes K$	-	Tensor product of H and K
$G' = [G, G]$	-	Derived subgroup of G
G/G'	-	The abelianization of G
$Z(G)$	-	Center of G
$x \in G$	-	x is an element of G
$x \notin G$	-	x is not an element of G
F^n	-	free group of rank n
F^n_{ab}	-	free abelian group of rank n
$\mathrm{h}(G)$	-	Hirsch length of G

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CHAPTER 1

INTRODUCTION

1.1 Introduction

A free abelian group G is the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow 1 \longrightarrow 1$$

where the point group is finite (trivial). A crystallographic group G is a generalization of free abelian groups since it satisfies the short exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow P \longrightarrow 1$$

where P is a point group that is a finite group acting faithfully on a maximal normal free abelian subgroup L of G which is of finite rank. The subgroup L is called a lattice group. It follows that L is a Fitting subgroup of G and its rank or Hirsch length is referred to as the dimension of G. Crystallographic groups arise as discrete, irreducible subgroups of the group of isometries of the $n-$dimensional Euclidean space. They are used to study the structures and characteristics of crystals. One of the crystallographic point groups that the researchers had an interest is Bieberbach groups. Bieberbach groups are torsion-free crystallographic groups.

The nonabelian tensor square $G \otimes G$ of a group G is a special case of the nonabelian tensor product $G \otimes H$ for two arbitrary group G and H that was introduced by Brown and Loday [1] extending the idea of Whitehead [2]. The nonabelian tensor square $G \otimes G$ is a group generated by the symbols $g \otimes h$ for $g, h \in G$, subject to relations

$$gh \otimes k = (g^h \otimes k^h)(h \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes k)(g^k \otimes h^k)$$

for all $g, h, k \in G$ where $g^h = h^{-1}gh$. It is defined by a presentation on $|G|^2$ generators and $2|G|^3$ relations. This presentation does not reflect the group structure very well. The structure of $G \otimes G$ has been first studied by Brown, Johnson and Robertson [3] where they investigated the group structures in terms of central extensions. Their focus is to compute the nonabelian tensor square of groups by finding a simplified presentation for the nonabelian tensor square from the presentation given by the definition.

1.2 Research Background

Bieberbach groups are torsion-free crystallographic groups. They appear as fundamental groups of compact, connected, flat Riemannian manifolds and have many interesting algebraic properties [4, 5]. Any new properties or results concerning crystallographic groups, particularly Bieberbach groups might lead to new exploration of the groups by not only mathematicians but by physicists and chemists too. New properties and results of the groups can be obtained by, not limited to, exploring the groups and by computing their nonabelian tensor squares.

Eventhough the work on computing the nonabelian tensor squares of groups have started a long time ago, but the work on computing the nonabelian tensor squares of Bieberbach groups has just started by Rohaidah [6] in 2009. She computed the nonabelian tensor squares of Bieberbach groups with cyclic point groups. One of the main goals of this research is to compute the nonabelian tensor squares of Bieberbach groups with a nonabelian point group, particularly with a dihedral point group of order eight. The exploration of the groups in the

Crystallographic AlgoRithms And Tables (CARAT) [7] and the computation of the nonabelian tensor squares of the groups with the Groups, Algrorithms, and Programming (GAP) software [8] lead us first to characterize exactly the general Bieberbach group with trivial center. The centerless Bieberbach groups are of interest since they do not arise in the general construction of a Bieberbach group for any given finite point group. It will be shown that the construction depends on the presentation of the point group.

1.3 Problem Statement

To explore the properties of Bieberbach groups with any finite point group and to calculate the nonabelian tensor squares of Bieberbach groups with nonabelian point group; in particular, the dihedral point group of order eight.

1.4 Research Objectives

The objectives of this research are

(i) to construct a Bieberbach group with an arbitrary finite point group

(ii) to give examples of the construction of Bieberbach groups in (i) with a cyclic point group of order two and with a dihedral point group of order eight

(iii) to characterize Bieberbach groups with trivial center as exactly those with finite abelianization

(iv) to determine the properties of centerless Bieberbach groups with a dihedral point group of order eight and their nonabelian tensor squares

(v) to compute the nonabelian tensor squares of a centerless Bieberbach group of dimension four with dihedral point group of order eight and whose derived length of its nonabelian tensor squares is two and

(vi) to compute some homological functors of Bieberbach groups with dihedral point group of order eight and with other nonabelian point groups with GAP.

1.5 Scope of Thesis

In this thesis, only $n-$dimension Bieberbach groups with any point group are considered. Properties of only centerless Bieberbach groups with dihedral point group of order eight and their nonabelian tensor squares are obtained. In computing the nonabelian tensor squares of a group, a group is limited to a centerless Bieberbach group of dimension four with a dihedral point group of order eight in which the derived length of its nonabelian tensor squares is two.

1.6 Significance of Findings

The major contribution of this thesis will be new theoretical results on constructing and characterizing the Bieberbach groups with any finite point groups. The properties of the centerless Bieberbach groups of dimension four with dihedral point group of order eight and computing their nonabelian tensor squares will contribute as a foundation in determining the generalization of their nonabelian tensor squares and the nonabelian tensor squares of Bieberbach groups with other nonabelian finite point groups. Thus this thesis contributes to new findings in the field of theoretical and computational group theory.

1.7 Thesis Outline

There are eight chapters in this thesis. Chapter 1 presents the introduction of the thesis. This chapter discusses research background, problem statement, research objectives, research scope and the significance of the thesis.

In Chapter 2, the studies of crystallographic groups especially Bieberbach groups are presented. Some characterizations of Bieberbach groups given by several researchers are discussed. The background of the nonabelian tensor squares of groups are overviewed and the methods of computing the nonabelian tensor squares of various groups by several researchers are compared. The method of computing the nonabelian tensor squares initiated by Rocco [9] followed by Ellis and Leonard [10] and extended by Blyth and Morse [11] is presented briefly in this chapter. The background and the application of the software system for computational group theory, GAP, and a computer package consists of a library of Bieberbach groups, CARAT, are also presented in this chapter.

Chapter 3 is a chapter of preliminary results. It presents some related and important definitions in group theory that are used throughout the thesis. Some basic results on free groups are presented in this chapter. Methods chosen in computing the nonabelian tensor squares of polycyclic groups and related results that are used in the thesis are elaborated deeply in one of the section in this chapter. A list of commutator calculus is also given here for easy reference.

Chapter 4 discusses the main result on constructing a Bieberbach group with an arbitrary point group. The proof of the existence of a Bieberbach group with any finite point group are presented in this chapter. With the construction discussed, examples of Bieberbach groups with a cyclic point group of order two and a dihedral point group of order eight are given here.

Chapter 5 presents a new characterization of any Bieberbach group with finite point group. The characterization is based on the structure of the abelianization of a centerless Bieberbach group.

Chapter 6 discusses the theory and the calculation of the nonabelian tensor squares of centerless Bieberbach groups with a dihedral point group of order eight. The theory gives us the properties of the groups and its nonabelian tensor squares. In this chapter, with the method chosen, the computation of the nonabelian tensor squares of a centerless Bieberbach group of dimension four with a dihedral point group of order eight in which the derived length of its nonabelian tensor squares is two is presented. The polycyclic presentation of the group is shown to be consistent so that GAP can be used to assist the hand computation. The nonabelian tensor square of the group is presented by a simple presentation of its generators and relations.

The exploration of some homological functors of all centerless Bieberbach groups with a dihedral point group of order eight with GAP is discussed in Chapter 7. In this chapter the exploration is not limited to the centerless Bieberbach groups but also to Bieberbach groups with other nonabelian point groups. Examples of GAP codes for the purposes are presented in this chapter. Few results regarding some of the homological functors of the groups are also presented.

Lastly, the thesis is summarized in Chapter 8. Some suggestions for future research on the nonabelian tensor square of the Bieberbach groups and other homological functors of the groups are given in this chapter.

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